PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Reduction of Error by Linear Compounding

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Phil. Trans. R. Soc. Lond. A 1921 **221**, 199-237
doi: 10.1098/rsta.1921.0007

VII. *Reduction of Error by Linear Compounding.*

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Communicated by E. T. WHITTAKER, *F.R.S.*

Received April 23,—Read June 24, 1920.

CONTENTS.

APPENDICES.

1. *Introductory.*—This paper is a development of two earlier papers,* which for brevity I call "Reduction" and "Fitting" respectively. The paper† immediately preceding "Fitting" is referred to as "Factorial Moments."

These earlier papers deal with two problems, which are closely connected and have the same solution. For both of them, the data are a set of quantities u_0, u_1, u_2, ... of the same kind, which we regard as representing certain true values U_0, U_1, U_2, ..., with errors e_0, e_1, e_2, ..., so that $u_r = U_r + e_r$. These errors may be independent or may be correlated in any way. The first problem is based on the assumption (which defines the class of cases we are dealing with) that the sequence of U's is fairly regular, so that differences after those of a certain order, which we will call j, are negligible. This being so, we may alter any u, or any linear compound of the u's, such as an interpolation-formula, by adding to it any linear compound of the negligible differences. (I use the term "linear compound" in preference to "linear function," since there is no consideration of functionality.) The problem is to find the value of the resulting sum when, by suitable choice of the coefficients in the added portion, the mean square of error of the sum is a minimum. This is the problem of "reduction of error." For the second problem it is assumed that U_r is a polynomial in r of degree j, and the problem is to find the coefficients in this polynomial by the method of least squares. This is the problem of "fitting."

The practical solution of these problems for the general case, in which the errors are correlated, is not easy. The particular case which is simple is that in which the errors all have the same mean square, which by a suitable choice of unit is taken to be 1, and the mean products of error are all 0. (In the previous papers I have called this system of errors the *standard* system; in the present paper the set of u's which possesses this property is called a *self-conjugate* set.) In "Reduction" I gave the solution for this particular case in terms of central differences, and in "Fitting" I gave the solution in terms of advancing differences and of advancing and central sums, formed in a particular way. I also gave expressions in terms of the u's, but these were rather complicated. It remained to obtain expressions for the mean squares of error of the new values, in order to compare them with those of the old

* "Reduction of Errors by means of Negligible Differences," 'Fifth International Congress of Mathematicians,' Cambridge, 1912, ii., 348–384; "Fitting of Polynomial by Method of Least Squares," 'Proceedings of the London Mathematical Society,' 2nd series, xiii., 97–108.

† "Factorial Moments in terms of Sums or Differences," 'Proceedings of the London Mathematical Society,' 2nd series, xiii., 81–96.

values. In doing this I found that the whole of the work could be very much simplified by using certain general theorems, which applied not only to the special case of the standard system but also to the general case, and even to a still more general problem in which, in the one aspect, the reduction of error is effected by means of quantities which are not necessarily a set of differences, or in which, in the other aspect, U_r is not necessarily a polynomial in r of degree j but is a linear compound, with coefficients to be determined, of any $j+1$ functions of r; and the present paper is mainly concerned with these general theorems, so that to a certain extent it supersedes the previous papers.

The abbreviations l.c., m.s.e., m.p.e., are used for linear compound, mean square of error, mean product of errors. The mean square of error of A is denoted by $(A\,;A)$, and the mean product of errors of A and B by $(A\,;B)$ or $(B\,;A)$. Other special notations used in the paper are the same as in the three papers mentioned at the beginning of this section, or are explained in §§ 3, 5 (iii.), 7, 17, and 20.

Conjugate Sets.

2. *Conjugate Set.*—(i.) Let $A, B, C, D, \ldots$ be a set of quantities, not necessarily all of the same kind, containing coexistent errors which are either independent or correlated in any way. For the purpose of the following investigations it is convenient to consider, in connexion with these quantities, another set of quantities, $G, H, J, K, \ldots$, equal to them in number and connected with them by the conditions that (1) each quantity of the second set is a l.c. of those of the first set, and (2) the m.p.e. of corresponding members of the two sets is 1 and that of members which do not correspond is 0. If we replace the quantities of the two sets by $A_0, A_1, A_2, \ldots$, and $G_0, G_1, G_2, \ldots$, we can express this latter condition by saying that m.p.e. of G_r and $A_s = 0$ $(s \neq r)$ or 1 $(s = r)$. The second set of quantities is said to be *conjugate* to the first.

(ii.) Let the member of the second set which corresponds to C of the first set be J. To determine J, let us write

$$J = aA+bB+cC+dD+\ldots.$$

Then, denoting the m.p.e. of A and B by $(A\,;B)$, condition (2) gives

$$\begin{aligned}(A\,;A)\,a+(A\,;B)\,b+(A\,;C)\,c+(A\,;D)\,d+\ldots &= 0,\\ (B\,;A)\,a+(B\,;B)\,b+(B\,;C)\,c+(B\,;D)\,d+\ldots &= 0,\\ (C\,;A)\,a+(C\,;B)\,b+(C\,;C)\,c+(C\,;D)\,d+\ldots &= 1,\\ (D\,;A)\,a+(D\,;B)\,b+(D\,;C)\,c+(D\,;D)\,d+\ldots &= 0,\end{aligned}$$

&c.

There are as many equations as there are coefficients $a, b, c, d, \ldots$; and the values of these are thus uniquely determined.

(iii.) The values of $a, b, c, \ldots$ as found from the above equations have as their denominator the determinant

$$\Theta \equiv \begin{vmatrix} (A\,;A) & (A\,;B) & (A\,;C) & (A\,;D) & \ldots \\ (B\,;A) & (B\,;B) & (B\,;C) & (B\,;D) & \ldots \\ (C\,;A) & (C\,;B) & (C\,;C) & (C\,;D) & \ldots \\ (D\,;A) & (D\,;B) & (D\,;C) & (D\,;D) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix}$$

There is therefore no conjugate set if this determinant is zero. The nature of the relations which in this case hold between the errors is considered in Appendix I., § 3.

(iv.) Since the members of the conjugate set are l.cc. of those of the original set, the converse also holds. Regrouping the equations which determine the coefficients, it will be seen that the original set is conjugate to the conjugate set; *i.e.*, that the two sets are conjugate to each other. The formulæ for the members of the original set in terms of those of the conjugate set are

$$\left.\begin{aligned} A &= (A\,;A)\,G+(A\,;B)\,H+(A\,;C)\,J+\ldots, \\ B &= (B\,;A)\,G+(B\,;B)\,H+(B\,;C)\,J+\ldots, \\ C &= (C\,;A)\,G+(C\,;B)\,H+(C\,;C)\,J+\ldots, \\ &\&c. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

These follow from the solution of the equations in (ii.), by the ordinary properties of determinants; or they may be obtained more simply by determining the coefficients of G, H, J, ... in each case from the second of the conditions stated in (i.).

(v). By means of these relations we can not only express any l.c. of the quantities of either set in terms of those of the other set, but we can also express any such l.c. in terms of particular quantities of one set and those of the conjugate set which correspond to the remaining quantities. We can, for instance, express any l.c. of $G, H, J, K, \ldots$ in terms of $A, B, J, K, \ldots$ by using the first two equations in (1) to determine G and H in terms of $A, B, J, K, \ldots$. The results involve a certain determinant in the denominator; it is shown in Appendix I., § 4, that this is not zero if Θ is not zero.

(vi.) Two special cases may be mentioned:—

(a) If the errors of $A, B, C, D, \ldots$ form a standard system, *i.e.*, if the m.s.e. of each of the quantities is 1 and the m.p.e. of each pair of quantities is 0, the conjugate set is identical with the original set; and conversely. A set which is identical with the conjugate set will be called a *self-conjugate* set.

(*b*) If the m.p.e. of each pair of quantities of the original set is 0, but the m.ss.e. are not all 1, this is also the case for the conjugate set. The original set being $A, B, C, \ldots$, the quantities of the conjugate set are $A/(A\,;A)$, $B/(B\,;B)$, $C/(C\,;C)$, ... ; and their m.ss.e. are $1/(A\,;A)$, $1/(B\,;B)$, $1/(C\,;C)$, ...

3. *Relations between Original Set and Conjugate Set.*—For expressing a member of either set in terms of the members of the other set, it is convenient to give them a linear order. We therefore denote the members of the original set by $\delta_0, \delta_1, \delta_2, \ldots \delta_l$ and those of the conjugate set by $\sigma_0, \sigma_1, \sigma_2, \ldots \sigma_l$. Also we write

$$\zeta_{r,t} \equiv \text{m.p.e. of } \delta_r \text{ and } \delta_t = \zeta_{t,r}, \qquad (2)$$

$$\eta_{r,t} \equiv \text{m.p.e. of } \sigma_r \text{ and } \sigma_t = \eta_{t,r}. \qquad (3)$$

(i.) The condition of conjugacy is that ($r = 0, 1, 2, \ldots l$; $t = 0, 1, 2, \ldots l$)

$$\text{m.p.e. of } \delta_r \text{ and } \sigma_t = 0\ (r \neq t) \quad \text{or} \quad 1\ (r = t). \qquad (4)$$

(ii.) The expression for δ_r in terms of the σ's is (*cf.* § 2 (iv.))

$$\delta_r = \zeta_{r,0}\sigma_0 + \zeta_{r,1}\sigma_1 + \zeta_{r,2}\sigma_2 + \ldots + \zeta_{r,l}\sigma_l. \qquad (5)$$

[For, if we write

$$\delta_r = \alpha_0\sigma_0 + \alpha_1\sigma_1 + \alpha_2\sigma_2 + \ldots + \alpha_l\sigma_l,$$

then (2) and (4) give

$$\begin{aligned}\zeta_{r,t} &= \text{m.p.e. of } \delta_t \text{ and } \alpha_0\sigma_0 + \alpha_1\sigma_1 + \ldots + \alpha_l\sigma_l \\ &= \alpha_t.]\end{aligned}$$

(iii.) Similarly the expression for σ_t in terms of the δ's is

$$\sigma_t = \eta_{0,t}\delta_0 + \eta_{1,t}\delta_1 + \eta_{2,t}\delta_2 + \ldots + \eta_{l,t}\delta_l. \qquad (6)$$

(iv.) The relations between the ζ's and the η's are easily deduced from (5) and (6). If we write

$$Z \equiv \begin{vmatrix} \zeta_{0,0} & \zeta_{0,1} & \zeta_{0,2} & \ldots & \zeta_{0,l} \\ \zeta_{1,0} & \zeta_{1,1} & \zeta_{1,2} & \ldots & \zeta_{1,l} \\ \zeta_{2,0} & \zeta_{2,1} & \zeta_{2,2} & \ldots & \zeta_{2,l} \\ \vdots & \vdots & \vdots & & \vdots \\ \zeta_{l,0} & \zeta_{l,1} & \zeta_{l,2} & \ldots & \zeta_{l,l} \end{vmatrix}, \qquad (7)$$

$$Z_{p,q} \equiv \text{cofactor of } \zeta_{p,q} \text{ in } Z = Z_{q,p}, \qquad (8)$$

$$H \equiv \begin{vmatrix} \eta_{0,0} & \eta_{0,1} & \eta_{0,2} & \cdots & \eta_{0,l} \\ \eta_{1,0} & \eta_{1,1} & \eta_{1,2} & \cdots & \eta_{1,l} \\ \eta_{2,0} & \eta_{2,1} & \eta_{2,2} & \cdots & \eta_{2,l} \\ \vdots & \vdots & \vdots & & \vdots \\ \eta_{l,0} & \eta_{l,1} & \eta_{l,2} & \cdots & \eta_{l,l} \end{vmatrix}, \quad \ldots \quad (9)$$

$$H_{p,q} \equiv \text{cofactor of } \eta_{p,q} \text{ in } H = H_{q,p}, \quad \ldots \quad (10)$$

then

$$\eta_{p,q} = Z_{p,q}/Z, \quad \ldots \quad (11)$$

$$\zeta_{p,q} = H_{p,q}/H, \quad \ldots \quad (12)$$

$$HZ = 1. \quad \ldots \quad (13)$$

(v.) The assumption that there is a conjugate set implies (*cf.* § 2 (iii.)) that Z is not = 0. It follows from (13) that H is not = 0. It also follows (see Appendix I., § 4 (*b*)) that none of the principal minors of Z or of H are = 0.

4. *Two Related Pairs of Conjugate Sets.*—(i.) Suppose that there is another set of $l+1$ quantities $u_0, u_1, u_2, \ldots u_l$, connected with the δ's by the linear relations $(r = 0, 1, 2, \ldots l)$

$$u_r = (r_0)\,\delta_0 + (r_1)\,\delta_1 + (r_2)\,\delta_2 + \ldots + (r_l)\,\delta_l. \quad \ldots \quad (14)$$

Then, by the condition of conjugacy of the δ's and the σ's,

$$(r_t) = \text{m.p.e. of } u_r \text{ and } \sigma_t. \quad \ldots \quad (15)$$

Let the set conjugate to $u_0, u_1, u_2, \ldots u_l$ be $y_0, y_1, y_2, \ldots y_l$. Then there are linear relations between the y's and the u's and between the σ's and the δ's, and therefore also, by (14), between the y's and the σ's. To find the σ's in terms of the y's, we write (15) in the form

$$(r_t) = \text{m.p.e. of } \sigma_t \text{ and } u_r\,;$$

and we see that $(t = 0, 1, 2, \ldots l)$

$$\sigma_t = (0_t)\,y_0 + (1_t)\,y_1 + (2_t)\,y_2 + \ldots + (l_t)\,y_l. \quad \ldots \quad (16)$$

(ii.) Similarly, if the expression for the δ's in terms of the u's is $(s = 0, 1, 2, \ldots l)$

$$\delta_s = \{s_0\}\,u_0 + \{s_1\}\,u_1 + \{s_2\}\,u_2 + \ldots + \{s_l\}\,u_l, \quad \ldots \quad (17)$$

where, by (14),

$$(r_0)\,\{0_t\} + (r_1)\,\{1_t\} + \ldots + (r_l)\,\{l_t\} = 0\ (r \neq t) \quad \text{or} \quad 1\ (r = t), \quad \ldots \quad (18)$$

$$\{s_0\}\,(0_t) + \{s_1\}\,(1_t) + \ldots + \{s_l\}\,(l_t) = 0\ (r \neq t) \quad \text{or} \quad 1\ (r = t), \quad \ldots \quad (19)$$

then

$$\{s_t\} = \text{m.p.e. of } \delta_s \text{ and } y_t, \quad \ldots \quad (20)$$

$$y_t = \{0_t\}\,\sigma_0 + \{1_t\}\,\sigma_1 + \{2_t\}\,\sigma_2 + \ldots + \{l_t\}\,\sigma_l. \quad \ldots \quad (21)$$

(iii.) The above relations can be expressed diagrammatically, thus :—

⋮	⋮	⋮	⋮	⋮		⋮
δ_3	×	×	×	×	. . .	σ_3
δ_2	×	×	×	×	. . .	σ_2
δ_1	×	×	×	×	. . .	σ_1
δ_0	×	×	×	×	. . .	σ_0
	u_0	u_1	u_2	u_3	. . .	
	y_0	y_1	y_2	y_3	. . .	

The crosses represent the ( ) coefficients if they are the coefficients of the δ's in the u's and of the y's in the σ's, and the { } coefficients in the converse case.

(iv.) Similarly, if we write ($r = 0, 1, 2, \ldots l$; $t = 0, 1, 2, \ldots l$)

$$[r_t] \equiv \text{m.p.e. of } y_r \text{ and } \sigma_t, \quad \ldots \ldots \ldots \quad (22)$$

then

$$y_r = [r_0]\,\delta_0 + [r_1]\,\delta_1 + [r_2]\,\delta_2 + \ldots + [r_l]\,\delta_l, \quad \ldots \ldots \quad (23)$$

$$\sigma_t = [0_t]\,u_0 + [1_t]\,u_1 + [2_t]\,u_2 + \ldots + [l_t]\,u_l. \quad \ldots \ldots \quad (24)$$

5. *Sums as Conjugates of Differences.*—The cases of importance are those in which the δ's are successive differences of the u's. It will be found that in these cases the σ's are l.cc. of successive sums of the y's.

(i.) Let the δ's be the advancing differences of the u's, *i.e.*

$$\delta_0 = u_0, \; \delta_1 = \Delta u_0, \ldots \delta_r = \Delta^r u_0, \ldots.$$

Then the diagram for the ( ) coefficients is

⋮	⋮	⋮	⋮	⋮		⋮
δ_3	0	0	0	1	. . .	σ_3
δ_2	0	0	1	3	. . .	σ_2
δ_1	0	1	2	3	. . .	σ_1
δ_0	1	1	1	1	. . .	σ_0
	u_0	u_1	u_2	u_3	. . .	
	y_0	y_1	y_2	y_3	. . .	

so that

$$\sigma_0 = y_0+y_1+y_2+y_3+\ldots+y_l,$$

$$\sigma_1 = y_1+2y_2+3y_3+4y_4+\ldots+ly_l,$$

$$\sigma_2 = y_2+3y_3+6y_4+10y_5+\ldots+\tfrac{1}{2}l\,(l-1)\,y_l,$$

and, generally,

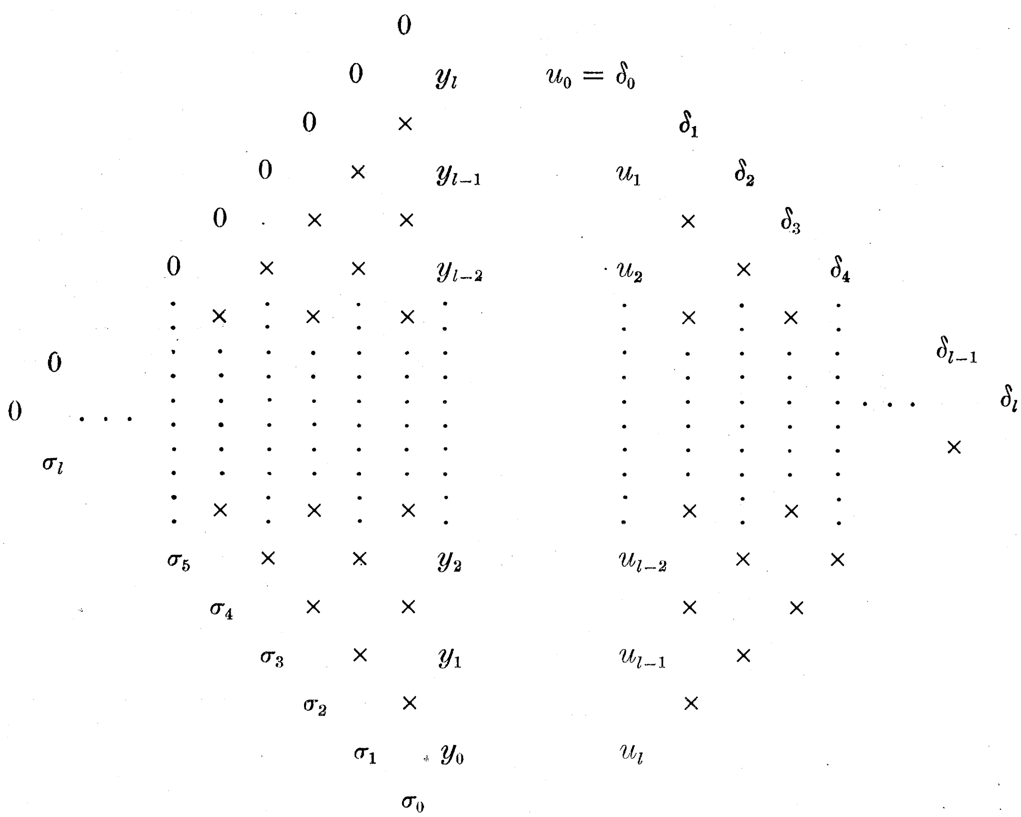
$$\sigma_f = y_f+(f+1,\,1)\,y_{f+1}+(f+2,\,2)\,y_{f+2}+\ldots+(l,\,l-f)\,y_l$$

$$= \sum_{q=f}^{q=l}(q,f)\,y_q\,; \quad \ldots \ldots \ldots \ldots \ldots \quad (25)$$

or, in the notation of "Fitting," § 4, and "Factorial Moments,"

$$\sigma_f = \Sigma''^{f+1}y_f. \quad \ldots \ldots \ldots \ldots \quad (26)$$

The σ's can be obtained by successive summations of the y's in reverse order, *i.e.* from y_l to y_0, as shown in the following diagram, in which the crosses represent entries in a sum- or difference-table :—



(ii.) Let the δ's be the central differences of the u's. Then it will be found in the same way that

(a) If the u's are $u_0, u_1, u_2, \ldots u_{2n}$, so that

$$\delta_0 = u_n,\ \delta_1 = \mu\delta u_n,\ \delta_2 = \delta^2 u_n,\ \delta_3 = \mu\delta^3 u_n,\ \ldots,$$

then

$$\sigma_{2h} = \sum_{r=-n}^{r=n} [r, 2h)\, y_{n+r}, \qquad (27)$$

$$\sigma_{2h-1} = \sum_{r=-n}^{r=n} (r, 2h-1]\, y_{n+r}; \qquad (28)$$

(*b*) If the u's are $u_0, u_1, u_2, \ldots u_{2n-1}$, so that

$$\delta_0 = \mu u_{n-\frac{1}{2}},\ \delta_1 = \delta u_{n-\frac{1}{2}},\ \delta_2 = \mu\delta^2 u_{n-\frac{1}{2}},\ \delta_3 = \delta^3 u_{n-\frac{1}{2}},\ \ldots,$$

then

$$\sigma_{2h-1} = \sum_{r=-n+1}^{r=n} [r-\tfrac{1}{2}, 2h-1)\, y_{n+r-1}, \qquad (29)$$

$$\sigma_{2h-2} = \sum_{r=-n+1}^{r=n} (r-\tfrac{1}{2}, 2h-2]\, y_{n+r-1}. \qquad (30)$$

(iii.) The values given by (25)–(30) may be expressed in terms of successive sums by the formulæ given in "Factorial Moments." The notation, however, can be simplified. Suppose that we have a set of quantities $\ldots F_0, F_1, F_2, \ldots$ corresponding to values $\ldots 0, 1, 2, \ldots$ of some variable, and that we form the table of successive differences (and also, if we like, of successive sums) of the F's. Then the Lagrangian formula for F_θ in terms of $F_p, F_{p+1}, \ldots F_{p+t}$, which can be expressed in a good many different ways, may be regarded as the formula for it in terms of the whole (unlimited) set of differences (and sums) which form a triangle with its apex at $\Delta^t F_p$; and we can denote it by $L\{F_\theta; \Delta^t F_p\}$. With this notation, the above results may be written

(25) $$\sigma_f = \left[L\{(-)^f \Sigma^{f+1} y_f; \Sigma y_t\}\right]_{t=0}^{t=l+1}, \qquad (31)$$

(27) $$\sigma_{2h} = \left[L\{\mu\sigma^{2h+1} y_n; \sigma y_{n+t}\}\right]_{t=-n-\frac{1}{2}}^{t=n+\frac{1}{2}}, \qquad (32)$$

(28) $$\sigma_{2h-1} = \left[L\{-\sigma^{2h} y_n; \sigma y_{n+t}\}\right]_{t=-n-\frac{1}{2}}^{t=n+\frac{1}{2}}, \qquad (33)$$

(29) $$\sigma_{2h-1} = \left[L\{-\mu\sigma^{2h} y_{n-\frac{1}{2}}; \sigma y_{n+t-1}\}\right]_{t=-n+\frac{1}{2}}^{t=n+\frac{1}{2}}, \qquad (34)$$

(30) $$\sigma_{2h-2} = \left[L\{\sigma^{2h-1} y_{n-\frac{1}{2}}; \sigma y_{n+t-1}\}\right]_{t=-n+\frac{1}{2}}^{t=n+\frac{1}{2}}. \qquad (35)$$

The L is distributive as regards the first member inside the { }; *e.g.*, in the case of (31),

$$A\sigma_2 + B\sigma_3 = \left[L\{A\Sigma^3 y_2 - B\Sigma^4 y_3; \Sigma y_t\}\right]_{t=0}^{t=l+1}.$$

(iv.) More generally, suppose that the δ's are the successive differences of the u's according to any system of differences; by which we mean that δ_s is either a definite difference of the u's of order s (the u's themselves being differences of order 0) or a l.c. of such differences. Then (r_t) of (14) is a polynomial in r of degree t, and σ_t is of the form $\sum_{q=0}^{q=l} \phi_t(q)\, y_q$, where $\phi_t(q)$ is some polynomial in q of degree t. It follows that any l.c. of $\sigma_0, \sigma_1, \sigma_2, \ldots \sigma_t$ is also of this form.

(v.) If we denote the f^{th} moment of the y's by M_f, then M_f is of the form $\sum_{q=0}^{q=m} \phi_f(q)\, y_q$. Hence σ_t is a l.c. of $M_0, M_1, M_2, \ldots M_t$; and M_t is a l.c. of $\sigma_0, \sigma_1, \sigma_2, \ldots \sigma_t$. More generally, any l.c. of $\sigma_0, \sigma_1, \sigma_2, \ldots \sigma_t$ is a l.c. of $M_0, M_1, M_2, \ldots M_t$; and conversely.

Reduction of Error (General).

6. *General Theorems.*—Let $A, B, C, \ldots P, Q, R, \ldots$ be a set of quantities as in § 2, but all of the same kind. If

$$w \equiv aA + bB + cC + \ldots \text{(with or without terms in } P, Q, R, \ldots),$$

$$x = w + pP + qQ + rR + \ldots,$$

where $a, b, c, \ldots$ are fixed and $p, q, r, \ldots$ are arbitrary, and if we choose $p, q, r, \ldots$ so as to make the m.s.e. of x a minimum, the resulting value of x is called the *improved value of* w, *using* P, Q, R, … *as auxiliaries.* The following are general theorems; some are quite elementary, but it is convenient to state them here. The specially important theorems are (III.) and (XIII.). The assumption mentioned under (VI.) should be noted. If strict proofs of (I.) and (II.) are required, the method should be that of Appendix I., § 2.

(I.) *The m.p.e. of* A *and any l.c. of* A, B, C, … *is the same l.c. of the m.pp.e. of* A *and* A, B, C, … [*i.e.*, m.p.e. of A and $aA + bB + cC + \ldots$ is $a(A\,;\,A) + b(A\,;\,B) + c(A\,;\,C) + \ldots$].

(II.) *The m.s.e. of any l.c. of* A, B, C, …, *or the m.p.e. of any two such l.cc., is found by squaring the former or multiplying the latter and replacing squares and products by the corresponding m.ss.e. and m.pp.e.* [*i.e.*, m.s.e. of $aA + bB + cC + \ldots$ $= a^2(A\,;\,A) + 2ab(A\,;\,B) + b^2(B\,;\,B) + 2ac(A\,;\,C) + 2bc(B\,;\,C) + c^2(C\,;\,C) + \ldots$, and similarly for m.p.e. of $aA + bB + cC + \ldots$ and $a'A + b'B + c'C + \ldots$].

(III.) *If the improved value of* A, *using certain auxiliaries, is* $A + \alpha$, *then the m.p.e. of* $A + \alpha$ *and each of the auxiliaries or* α *or any other l.c. of the auxiliaries is zero.* [Let the auxiliaries be $P, Q, R, \ldots$, and let $A + \alpha = A + pP + qQ + rR + \ldots$. Then the m.s.e. of $A + (p+\theta)P + qQ + rR + \ldots (= A + \alpha + \theta P)$ is $(A+\alpha\,;\,A+\alpha) + 2\theta(A+\alpha\,;\,P) + \theta^2(P\,;\,P)$. In order that this may be a minimum for $\theta = 0$, $(A+\alpha\,;\,P)$ must be zero. Similarly for $(A+\alpha\,;\,Q)$, $(A+\alpha\,;\,R)$, …. This proves the first part of the theorem; the second then follows from (I.).] Hence

(IV.) *If the improved values of* A *and of* B, *using in each case the same set of auxiliaries, are* $A + \alpha$ *and* $B + \beta$, *the m.pp.e. of* $A + \alpha$ *and* $B + \beta$, *of* $A + \alpha$ *and* B, *and of* A *and* $B + \beta$, *are all equal;* and

(V.) *If the improved value of* A, *using certain auxiliaries, is* A + α, *and that of* B, *using some only of these, is* B + β, *the m.p.e. of* A + α *and* B + β *is equal to that of* A + α *and* B.

(VI.) *If the improved value of* A, *using* P, Q, R, ... *as auxiliaries, is* A + pP + qQ + rR + ..., *the values of* p, q, r, ... *are given by a set of linear relations between* p, q, r, ... *and the m.pp.e. of* A *and* P, Q, R, ... [The equations are given by (III.), viz., since $(A + pP + qQ + rR + \ldots ; P) = 0$, &c.,

$$(A\ ; P) + p\,(P\ ; P) + q\,(Q\ ; P) + r\,(R\ ; P) + \ldots = 0,$$
$$(A\ ; Q) + p\,(P\ ; Q) + q\,(Q\ ; Q) + r\,(R\ ; Q) + \ldots = 0,$$
$$(A\ ; R) + p\,(P\ ; R) + q\,(Q\ ; R) + r\,(R\ ; R) + \ldots = 0,$$
&c.

It is assumed that the determinant

$$\begin{vmatrix} (P\ ; P) & (Q\ ; P) & (R\ ; P) & \ldots \\ (P\ ; Q) & (Q\ ; Q) & (R\ ; Q) & \ldots \\ (P\ ; R) & (Q\ ; R) & (R\ ; R) & \ldots \\ \vdots & \vdots & \vdots & \end{vmatrix}$$

is not zero; *i.e.* (see § 2 (iii.)) that there is a set conjugate to $P, Q, R, \ldots$.]

(VII.) *For any particular set of auxiliaries there is one and only one improved value of* A. [This follows from the fact that the equations in (VI.), on the assumption there stated, have one and only one solution.] Hence we get the converse of (III.):—

(VIII.) *If* x *is the sum of* w *and a l.c. of* P, Q, R, ..., *and if the m.p.e. of* x *and each of* P, Q, R, ... *is zero, then* x *is the improved value of* w, *using* P, Q, R, ... *as auxiliaries.* As a particular case:—

(IX.) *If the m.p.e. of* w *and each of* P, Q, R, ... *is zero, the improved value of* w, *using* P, Q, R, ... *as auxiliaries, is the same as the original value.*

(X.) *The improved value of* P, *using* P, Q, R, ... *as auxiliaries, is* P − P = 0. [This follows either from (VIII.) or from taking $A = P$ in the equations in (VI.).] Hence

(XI.) *If* w *is a l.c. of* A, B, C, ... P, Q, R, ..., *the improved value of* w, *using* P, Q, R, ... *as auxiliaries, is the same as that of the quantity obtained by adding to* w *any l.c. of* P, Q, R, ...; and, conversely:—

(XII.) *If the improved values of* w *and of* w′, *using in each case the same set of auxiliaries, are identical, then* w *and* w′ *either are identical or differ by a l.c. of the auxiliaries.*

(XIII.) *If the improved values of* A, B, C, ..., *using in each case the same set of* *auxiliaries, are* A + α, B + β, C + γ, ... , *the improved value of any l.c. of* A, B, C, ... , *using these auxiliaries, is the same l.c. of* A + α, B + β, C + γ, ... [Let the l.c. of A, B, C, ... be $w \equiv aA + bB + cC + \ldots$. We want to prove that $x \equiv a(A+\alpha) + b(B+\beta) + c(C+\gamma) + \ldots$ is its improved value. We can do this in either of two ways:

(i.) By (III.), the m.p.e. of x and each of the auxiliaries P, Q, R, ... is zero; and x differs from w by a l.c. of P, Q, R, ... Hence, by (VIII.), x is the improved value of w, using P, Q, R, ... as auxiliaries.

(ii.) A more direct proof follows from the linearity of the equations mentioned in (VI.). It is not necessary to set out the proof here.]

(XIV.) *If* A, B, ... C, D, ... P, Q, ... R, S, ... *fall into two classes* A, B, ... P, Q, ... *and* C, D, ... , R, S, ... , *such that the m.p.e. of each member of the one class and each member of the other class is zero, then the improved value of a l.c. of any of the members, using* P, Q, ... R, S, ... *as auxiliaries, is to be found by taking the two classes separately, i.e.*, by using P, Q, ... as auxiliaries for the terms in A, B, ... P, Q, ..., and R, S, ... as auxiliaries for the terms in C, D, ... R, S, ... [For the m.s.e. of $aA + bB + \ldots + cC + dD + \ldots + pP + qQ + \ldots + rR + sS + \ldots$ is the sum of those of $aA + bB + \ldots + pP + qQ + \ldots$ and $cC + dD + \ldots + rR + sS + \ldots$, since the m.p.e. of these latter is zero; we cannot reduce the m.s.e. of the first of them by adding terms in R, S, ... , or that of the second of them by adding terms in P, Q, ... ; and the result is therefore the same as if we considered them separately.]

(XV.) *If the improved value of* w, *using* P, Q, R, ... *as auxiliaries, is* x = w + pP + *terms in* Q, R, ..., *this is also the improved value of* w + pP, *using* Q, R, ... *as auxiliaries.* [For x differs from $w + pP$ by terms in Q, R, ..., and the m.p.e. of x and each of Q, R, ... is zero.] This can be stated more generally as follows:—

(XVI.) *If the improved value of* A, *using a set of auxiliaries* S, *is* A + α, *and if we divide* S *into two sets*, S_1 *and* S_2, *and the corresponding parts of* α *are* α_1 *and* α_2, *then* A + α *is the improved value of* A + α_1, *using* S_2 *as auxiliaries.* [We may take ... P, Q to be S_1, and R, ... to be S_2. The theorem states that, if the improved value of A, using ... P, Q, R, ..., is $A + \ldots + pP + qQ + rR + \ldots$, this is also the improved value of $A + \ldots + pP + qQ$, using R, ...]

(XVII.) The following corollaries of (III.) may be noticed, though we shall not require them. If the improved values of A and of B, using in each case the same set of auxiliaries, are $A + \alpha$ and $B + \beta$, then

$$(1)\ (A+\alpha\,;\ A+\alpha) = (A\,;\ A) - (\alpha\,;\ \alpha)$$

and

$$(2)\ (A+\alpha\,;\ B+\beta) = (A\,;\ B) - (\alpha\,;\ \beta).$$

7. *Notation: and Particular Values.*—(i.) It will now be convenient to adopt a linear arrangement of the quantities we are dealing with, and we therefore replace

A, B, C, ... P, Q, R, ... by δ_0, δ_1, δ_2, ... δ_{j+1}, δ_{j+2}, ... δ_l. The order is quite arbitrary, so far as any general theorems are concerned; but it will usually be convenient to place the auxiliaries last. If, for instance, we are using all but $j+1$ as auxiliaries, we denote those not so used by δ_0, δ_1, δ_2, ... δ_j, and the auxiliaries by δ_{j+1}, δ_{j+2}, ... δ_l; the improved values are then denoted by $(\ )_j$.

(ii.) We use the following notation :—

$$(\epsilon_f)_j \equiv \text{improved value of } \delta_f, \text{ using } \delta\text{'s after } \delta_j;$$

$$\mathrm{E}_j \equiv (\epsilon_j)_j = \text{improved value of } \delta_j, \text{ using all subsequent } \delta\text{'s};$$

$$(\lambda_{f,g})_j \equiv \text{m.p.e. of } (\epsilon_f)_j \text{ and } (\epsilon_g)_j;$$

$$\Lambda_j \equiv (\lambda_{j,j})_j = \text{m.s.e. of } \mathrm{E}_j.$$

Where there is no doubt as to the δ's used as auxiliaries, $(\epsilon_f)_j$ and $(\lambda_{f,g})_j$ can be replaced by ϵ_f and $\lambda_{f,g}$.

(iii.) By (X.) of §6—

$$(\epsilon_f)_j = 0 \text{ if } f > j; \qquad (36)$$

$$(\lambda_{f,g})_j = 0 \text{ if } f > j \text{ or } g > j. \qquad (37)$$

(iv.) By (IV.)—

$$\left.\begin{aligned}(\lambda_{f,g})_j &= \text{m.p.e. of } (\epsilon_f)_j \text{ and } \delta_g \\ &= \text{m.p.e. of } \delta_f \text{ and } (\epsilon_g)_j\end{aligned}\right\}; \qquad (38)$$

$$\Lambda_j = \text{m.p.e. of } \mathrm{E}_j \text{ and } \delta_j. \qquad (39)$$

8. *Improved Values in terms of Conjugates.*—In "Fitting" I have given some formulæ for improved values in terms of sums. These may be regarded as derivable from a general theorem relating to the expression of improved values in terms of members of the conjugate set. The theorem is given by (XIX.) and (XX.) below; (XVIII.) is a particular case.

(i.) Take any one of the δ's as δ_0. By (6)—

$$\sigma_0 = \eta_{0,0}\delta_0 + \eta_{1,0}\delta_1 + \ldots + \eta_{l,0}\delta_l.$$

Hence $\sigma_0/\eta_{0,0}$ differs from δ_0 by a l.c. of the other δ's. Also the m.p.e. of $\sigma_0/\eta_{0,0}$ and each of these other δ's is zero. It follows from (VIII.) of §6 that $\sigma_0/\eta_{0,0}$ is the improved value of δ_0, using the other δ's as auxiliaries. The m.s.e. of this improved value is $\eta_{0,0}/(\eta_{0,0})^2 = 1/\eta_{0,0}$. Hence—

(XVIII.) *The improved value of any member of the original set, taking all the other members as auxiliaries, is the product of the corresponding member of the conjugate set by a constant; this constant being = the m.s.e. of the improved value.*

(ii.) The first part of the more general theorem is :—

(XIX.) *The improved value of any l.c. of a set of quantities, using those after the first* j+1 *as auxiliaries, is a l.c. of the first* j+1 *of the conjugate set.*

For, if w is a l.c. of $\delta_0, \delta_1, \delta_2, \ldots \delta_l$, we can (see §2 (v.)) express it as a l.c. of $\sigma_0, \sigma_1, \sigma_2, \ldots \sigma_j, \delta_{j+1}, \delta_{j+2}, \ldots \delta_l$. Let the result be $(\Sigma)+(\Delta)$, where (Σ) is a l.c. of $\sigma_0, \sigma_1, \sigma_2, \ldots \sigma_j$, and (Δ) is a l.c. of $\delta_{j+1}, \delta_{j+2}, \ldots \delta_l$ Then (Σ) differs from w by a l.c. of these latter δ's, and the m.p.e. of (Σ) and each of these δ's is 0; hence, by (VIII.), (Σ) is the improved value of w, using these δ's as auxiliaries.

(iii.) Further—

(XX.) *The coefficients of the σ's in the improved value of the l.c. are the m.pp.e. of this improved value and the corresponding δ's.*

For, if the improved value of w is x, and we write

$$x = b_0\sigma_0 + b_1\sigma_1 + b_2\sigma_2 + \ldots + b_j\sigma_j,$$

then, by the condition of conjugacy of the σ's and the δ's,

$$\text{m.p.e. of } x \text{ and } \delta_f = b_f.$$

(iv.) This would give us a solution of the problem of finding the improved value, if we could find the m.pp.e. Ordinarily, w is or can be expressed in terms of the δ's, and we do not find its improved value independently, but deduce it from those of the δ's up to δ_j. The improved value of δ_h is, by (iii),

$$(\epsilon_h)_j = (\lambda_{0,h})_j\sigma_0 + (\lambda_{1,h})_j\sigma_1 + \ldots + (\lambda_{j,h})_j\sigma_j\,; \quad \ldots \ldots \quad (40)$$

and the m.pp.e. that we really require are therefore the values of $(\lambda_{f,h})_j$. With regard to this, see §9.

(v.) As the converse of (XIX.) it may be noted that—

(XXI.) *A quantity of the conjugate set, or a l.c. of such quantities, cannot be improved by means of the non-corresponding quantities of the original set; e.g.,* a l.c. of σ_3 and σ_4 cannot be improved by using the δ's, other than δ_3 and δ_4, as auxiliaries. This follows from (IX.) of §6, since the m.p.e. of δ_r and σ_s is 0 unless $r = s$.

(vi.) If, as in § 5, there are related conjugate sets of u's and y's, and the δ's are the differences of the u's, it follows from § 5 (v.) that (Σ) in (ii.) above is a l.c. of the moments of the y's up to the j^{th}. (XIX.) is therefore a generalisation of the theorem, for a self-conjugate set, that the improved values are l.cc. of the moments; and, in fact, it explains the appearance of the moments in this connexion.

9. *Mean Products of Error of Improved Values.*—(i.) We have found, in § 8 (iv.), that

$$(\epsilon_h)_j = (\lambda_{0,h})_j\,\sigma_0 + (\lambda_{1,h})_j\,\sigma_1 + \ldots + (\lambda_{j,h})_j\,\sigma_j.$$

To obtain the λ's, we introduce the condition that this shall differ from δ_h by a l.c. of δ's after δ_j.

(ii.) Substituting for σ_0, σ_1, σ_2, ... from (6), this condition gives

$$\begin{aligned}&(\lambda_{0,h})_j(\eta_{0,0}\delta_0+\eta_{1,0}\delta_1+\ldots+\eta_{l,0}\delta_l)\\&+(\lambda_{1,h})_j(\eta_{0,1}\delta_0+\eta_{1,1}\delta_1+\ldots+\eta_{l,1}\delta_l)\\&+(\lambda_{2,h})_j(\eta_{0,2}\delta_0+\eta_{1,2}\delta_1+\ldots+\eta_{l,2}\delta_l)\\&+\ldots\\&+(\lambda_{j,h})_j(\eta_{0,j}\delta_0+\eta_{1,j}\delta_1+\ldots+\eta_{l,j}\delta_l)\\&=\delta_h+\text{ terms in }\delta_{j+1},\ \delta_{j+2},\ \ldots\ \delta_l.\end{aligned}$$

Equating the coefficients of δ_0, δ_1, δ_2, ... δ_j, we find that ($f = 0, 1, 2, \ldots j$)

$$\eta_{f,0}(\lambda_{0,h})_j+\eta_{f,1}(\lambda_{1,h})_j+\ldots+\eta_{f,j}(\lambda_{j,h})_j = 0\ (f \neq h) \text{ or } 1\ (f = h). \quad (41)$$

Let us write

$$\mathrm{H}_j \equiv \begin{vmatrix} \eta_{0,0} & \eta_{0,1} & \eta_{0,2} & \cdots & \eta_{0,j} \\ \eta_{1,0} & \eta_{1,1} & \eta_{1,2} & \cdots & \eta_{1,j} \\ \eta_{2,0} & \eta_{2,1} & \eta_{2,2} & \cdots & \eta_{2,j} \\ \vdots & \vdots & \vdots & & \vdots \\ \eta_{j,0} & \eta_{j,1} & \eta_{j,2} & \cdots & \eta_{j,j} \end{vmatrix}, \quad (42)$$

$$\mathrm{H}_{g,h,j} \equiv \text{cofactor of } \eta_{g,h} \text{ in } \mathrm{H}_j = \mathrm{H}_{h,g,j}. \quad (43)$$

Then

$$\eta_{f,0}\mathrm{H}_{0,h,j}+\eta_{f,1}\mathrm{H}_{1,h,j}+\ldots+\eta_{f,j}\mathrm{H}_{j,h,j} = 0\ (f \neq h) \text{ or } \mathrm{H}_j\ (f = h); \quad (44)$$

and therefore, by (41),

$$(\lambda_{g,h})_j = \mathrm{H}_{g,h,j}/\mathrm{H}_j. \quad (45)$$

Substituting in (40), we obtain $(\epsilon_h)_j$ in terms of the σ's.

For the particular case of $g = h = j$, $(\lambda_{g,h})_j$ becomes Λ_j, and $\mathrm{H}_{g,h,j}$ becomes H_{j-1}; so that

$$\Lambda_j = \mathrm{H}_{j-1}/\mathrm{H}_j. \quad (46)$$

(iii.) As an example, suppose that we have several independent observations, of unequal accuracy, of a single quantity U, and that we wish to obtain a suitably weighted mean, which may be regarded as the improved value of any one of the observations. Let the observed values be u_0, u_1, u_2, ..., and their m.ss.e. a_0^2, a_1^2, a_2^2, ...; the m.pp.e. being 0, since the observations are independent. We take δ_0 to be one of the u's, and δ_1, δ_2, ... to be its successive differences. Then $j = 0$, since the true values of the first and later differences are all 0. Hence, by § 8 (i.), the improved value is σ_0/(m.s.e. of σ_0). But, by § 5 (i.), $\sigma_0 = \Sigma y$; and, by § 2 (vi.) (b), $y_r = u_r/a_r^2$, so that m.s.e. of $\sigma_0 = \Sigma 1/a^2$. Hence the improved value is $\Sigma(u/a^2)/\Sigma(1/a^2)$.

(iv.) When j is relatively large, the solution given in (ii.) above can only be regarded as a formal one, since it involves calculation of determinants. I have not been able to provide a general solution which shall avoid determinants; but it will be seen in §§ 17–19 that, if we can find the values of certain quantities occurring in the formulæ, we can deduce the λ's and thence the coefficients of the σ's. These latter are important as giving us formulæ which contain only a few terms and are therefore suited for numerical calculation.

10. *Expressions in terms of a Related Set.*—Suppose that there is another set of $l+1$ quantities $u_0, u_1, u_2, \ldots u_l$, connected with the δ's by linear relations; and let the set conjugate to the u's be $y_0, y_1, y_2, \ldots y_l$. We shall take the relations between the u's and the δ's and between the y's and the δ's to be, as in § 4, $(r = 0, 1, 2, \ldots l)$

$$u_r = (r_0)\,\delta_0 + (r_1)\,\delta_1 + (r_2)\,\delta_2 + \ldots + (r_l)\,\delta_l, \quad (47)$$

$$y_r = [r_0]\,\delta_0 + [r_1]\,\delta_1 + [r_2]\,\delta_2 + \ldots + [r_l]\,\delta_l. \quad (48)$$

(i.) Let the improved value of u_r, using δ's after δ_j, be v_r. Then, from (47), by (XIII.), remembering that, by (36),

$$(\epsilon_f)_j = 0 \text{ if } f > j,$$

we have

$$v_r = (r_0)(\epsilon_0)_j + (r_1)(\epsilon_1)_j + \ldots + (r_l)(\epsilon_l)_j \quad (49)$$

$$= (r_0)(\epsilon_0)_j + (r_1)(\epsilon_1)_j + \ldots + (r_j)(\epsilon_j)_j. \quad (49\text{A})$$

Thus the v's are related to the ϵ's in the same way that the u's are related to the δ's.

(ii.) Similarly, if the improved value of y_r, using δ's after δ_j, is z_r, we have

$$z_r = [r_0](\epsilon_0)_j + [r_1](\epsilon_1)_j + \ldots + [r_l](\epsilon_l)_j \quad (50)$$

$$= [r_0](\epsilon_0)_j + [r_1](\epsilon_1)_j + \ldots + [r_j](\epsilon_j)_j; \quad (50\text{A})$$

and the z's are related to the ϵ's in the same way that the y's are related to the δ's.

(iii.) Let w be any l.c. of the δ's or of the u's or y's, and let x be its improved value, using δ's after δ_j. Suppose that x is expressed in terms of the y's, the coefficients being $p_0, p_1, p_2, \ldots p_l$, so that

$$x = p_0 y_0 + p_1 y_1 + p_2 y_2 + \ldots + p_l y_l; \quad (51)$$

and let

$$\lambda_g \equiv \text{m.p.e. of } x \text{ and } (\epsilon_g)_j,$$

so that

$$\lambda_g = 0 \text{ if } g > j.$$

Then, by the condition of conjugacy of the u's and the y's,

$$\text{m.p.e. of } x \text{ and } u_r = p_r. \quad (52)$$

Hence, by (IV.) and (49) or (49A),

$$p_r = \text{m.p.e. of } x \text{ and } v_r$$

$$= (r_0)\lambda_0 + (r_1)\lambda_1 + (r_2)\lambda_2 + \dots + (r_l)\lambda_l \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

$$= (r_0)\lambda_0 + (r_1)\lambda_1 + (r_2)\lambda_2 + \dots + (r_j)\lambda_j . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53\text{A})$$

Thus the p's are related to the λ's in the same way that the v's are related to the ϵ's, or the u's to the δ's.

(iv.) Similarly, if

$$x = q_0 u_0 + q_1 u_1 + q_2 u_2 + \dots + q_l u_l,$$

the q's are related to the λ's in the same way that the z's are related to the ϵ's, or the y's to the δ's.

11. *Special Case of Differences.*—The important practical case is that in which the δ's are successive differences of the u's, in the general sense explained in §5 (iv.). If the differences of order exceeding j are negligible, we can use them as auxiliaries for improving the u's or the δ's or any l.c. of the u's or the δ's.

(i.) Since the δ's are successive differences of the u's, (r_t) is (§5 (iv.)) a polynomial in r of degree t.

(ii.) By §10 (i.) the ϵ's are the differences of the v's according to the same system; and v_r is a polynomial of degree j in r, the differences of the v's of order exceeding j being zero.

(iii.) With the notation of §10 (iii.), the λ's are the differences of the p's according to the same system; and p_r is a polynomial of degree j in r, the differences of the p's of order exceeding j being zero.

(iv.) If we form the differences of the u's in the usual way, there will be l differences of order 1, $l-1$ of order 2, and so on. The $l-j+1$ of order j, namely $\Delta^j u_0, \Delta^j u_1, \dots \Delta^j u_{l-j}$, will differ from one another by l.cc. of the differences of higher order; and therefore, by (XI.), they will have the same improved value. If we denote this by E, then, if $w \equiv p\Delta^j u_0 + q\Delta^j u_1 + r\Delta^j u_2 + \dots$, the improved value of w is $(p+q+r+\dots)\text{E}$.

Relation of "Reduction of Error" to "Fitting" (of a Polynomial).

12. *Standard System.*—In the case of a standard system, the process of reduction of error and the process of fitting a polynomial (by least squares or by moments) give the same result. The following is a proof of this, not involving the properties of conjugate sets. The observed values are taken to be $u_0, u_1, u_2, \dots u_l$; and $\sum_t$ denotes summation for $t = 0, 1, 2, \dots l$.

(i.) If the polynomial which we are fitting to the u's is

$$v_q = a_0 + a_1 q + a_2 q^2 + \dots + a_j q^j, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

the values of the α's when we fit by least squares are given ("Fitting," §§1, 2) by the equations ($f = 0, 1, 2, \ldots j$)

$$\sum_q q^f . \alpha_0 + \sum_q q^{f+1} . \alpha_1 + \ldots + \sum_q q^{f+j} . \alpha_j = \sum_q q^f u_q \equiv \mathrm{M}_f . \quad . \quad . \quad . \quad . \quad (55)$$

These are the same equations that are given by the method of moments.

(ii.) The above equation (55) is a statement that the f^{th} moment of the v's is equal to that of the u's. In order to prove that the process of reduction of error, using differences of order exceeding j as auxiliaries, gives the same result, it is sufficient to show (α) that the improved value of u_q as given by this process is of the form of v_q in (54), and (β) that the f^{th} moment of the improved values of the u's is equal to that of the original values for $f = 0, 1, 2, \ldots j$.

(iii.) We have shown, in §11 (ii.), that the improved value of u_q is a polynomial of degree j in q. This establishes (α).

(iv.) By (XIII.), the f^{th} moment of the improved values of the u's is equal to the improved value of their f^{th} moment. In order to show that this is equal to the original value of the f^{th} moment, it is sufficient, by (IX.), to show that the m.p.e. of the original f^{th} moment and every difference of order exceeding j is zero.

(v.) Let the k^{th} difference of u_{r-k} be

$$\delta_k \equiv k_0 u_r - k_1 u_{r-1} + \ldots + (-)^k k_k u_{r-k}.$$

Then the f^{th} moment is

$$\ldots + r^f u_r + (r-1)^f u_{r-1} + (r-2)^f u_{r-2} + \ldots ,$$

and the m.p.e. of this and δ_k is

$$k_0 r^f - k_1 (r-1)^f + k_2 (r-2)^f - \ldots .$$

But this is the k^{th} difference of $(r-k)^f$, and is $= 0$ if $k > f$.

This proves the proposition.

13. *Fitting by Least Squares.*—Next suppose that the set is not self-conjugate. If the δ's were the differences of a set of u's, we should fit a polynomial of degree (say) j to the u's. This suggests that, in the more general case, the u's being connected with the δ's, as in § 10, by the relation ($r = 0, 1, 2, \ldots l$)

$$u_r = (r_0)\,\delta_0 + (r_1)\,\delta_1 + (r_2)\,\delta_2 + \ldots + (r_l)\,\delta_l, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

we should try to fit an expression of the form

$$v_r = (r_0)\,\epsilon_0 + (r_1)\,\epsilon_1 + (r_2)\,\epsilon_2 + \ldots + (r_j)\,\epsilon_j \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

to the u's by an appropriate method of least squares; the (r)'s being the same as in (56), and the ϵ's being the quantities to be determined.

(i.) If the y's are conjugate to the u's, and if

$$\psi_{r,s} \equiv \text{m.p.e. of } y_r \text{ and } y_s = \psi_{s,r}, \quad (58)$$

then (see Appendix II.) the direct (or *a priori*) probability of the occurrence of the given set of u's, if the v's as given by (57) were the true U's, is proportional to

$$\exp\{-\tfrac{1}{2}\sum_r\sum_s \psi_{r,s}(u_r-v_r)(u_s-v_s)\},$$

where $\sum_t$ denotes summation for $t = 0, 1, 2, \ldots l$. The principle of the method of least squares therefore leads us* to choose the ϵ's so as to make

$$\sum_r\sum_s \psi_{r,s}(u_r-v_r)(u_s-v_s)$$

a minimum. Differentiating with regard to each of the ϵ's, this gives ($f = 0, 1, 2, \ldots j$)

$$\sum_s \{(0_f)\psi_{0,s}+(1_f)\psi_{1,s}+\ldots+(l_f)\psi_{l,s}\}(v_s-u_s) = 0. \quad (59)$$

But, by (58) and (16),

$$(0_f)\psi_{0,s}+(1_f)\psi_{1,s}+\ldots+(l_f)\psi_{l,s} = \text{m.p.e. of } y_s \text{ and } (0_f)y_0+(1_f)y_1+\ldots+(l_f)y_l$$
$$= \text{m.p.e. of } y_s \text{ and } \sigma_f. \quad (60)$$

Denoting this, as in § 4 (iv.), by $[s_f]$, the equations (59) become ($f = 0, 1, 2, \ldots j$)

$$\sum_s [s_f](v_s-u_s) = 0. \quad (61)$$

(ii.) Instead of fitting an expression of the form given by (57) to the u's we might fit a corresponding expression to the y's. Since

$$y_s = [s_0]\delta_0+[s_1]\delta_1+[s_2]\delta_2+\ldots+[s_l]\delta_l, \quad (62)$$

the expression to be fitted would be of the form

$$z_s \equiv [s_0]\epsilon_0+[s_1]\epsilon_1+[s_2]\epsilon_2+\ldots+[s_j]\epsilon_j. \quad (63)$$

* Strictly, we ought to choose the ϵ's so as to make $B\,.\,C\exp-\tfrac{1}{2}P$ a maximum: where

$$P \equiv \Sigma\Sigma\psi_{r,s}(u_r-v_r)(u_s-v_s);$$

B is the direct probability of occurrence of the particular ϵ's denoted by $\epsilon_0, \epsilon_1, \epsilon_2, \ldots \epsilon_j$, and is therefore some function of these latter; and $C\exp-\tfrac{1}{2}P$ is the direct probability of occurrence of the particular values of $u-v$ on the assumption that these values of the ϵ's are the correct ones, C being some function of these ϵ's. But I have assumed, as is commonly done, that the range of practically possible values of the ϵ's is so small that B and C may be treated as constants, so that we have only to consider the maximum value of $\exp-\tfrac{1}{2}P$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

We should have to choose the ϵ's so as to make

$$\sum_r \sum_s \pi_{r,s} (y_r - z_r)(y_s - z_s)$$

a minimum, where

$$\pi_{r,s} \equiv \text{m.p.e. of } u_r \text{ and } u_s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

This would give

$$\sum_r (r_f)(z_r - y_r) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

(iii.) The ϵ's given by (65) are the same as are given by (61). For we have seen in § 4 that

$$(r_f) = \text{m.p.e. of } u_r \text{ and } \sigma_f, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

$$[s_f] = \text{m.p.e. of } y_s \text{ and } \sigma_f. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

If we express the u's in terms of the δ's, and write

$$\sum_s [s_f] u_s = \sum_t A_t \delta_t,$$

then, by the condition of conjugacy of the σ's and the δ's, and by (66),

$$A_t = \text{m.p.e. of } \sigma_t \text{ and } \sum_s [s_f] u_s$$
$$= \sum_s [s_f] (s_t).$$

This is symmetrical, and we should get the same expression for the coefficient of δ_t in $\sum_r (r_f) y_r$, so that

$$\sum_s [s_f] u_s = \sum_r (r_f) y_r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

Similarly, if we substitute the values of v_s from (57) and of z_r from (63) in $\sum_s [s_f] v_s$ and in $\sum_r (r_f) z_r$, the coefficients of ϵ_t in the resulting expressions are equal. Hence (61) and (65) are identical equations in the ϵ's.

(iv.) The values of the ϵ's as given by these equations are in fact independent of the u's or the y's. For the value of A_t as found in (iii.) above is

$$\sum_s [s_f] (s_t) = \text{m.p.e. of } \sum_s [s_f] u_s \text{ and } \sum_s (s_t) y_s$$
$$= \text{m.p.e. of } \sigma_f \text{ and } \sigma_t, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

by (24) and (16). Hence, denoting the m.p.e. of σ_f and σ_t, as in § **3**, by $\eta_{f,t}$, the ϵ's given by (61) or (65) are the same as would be given by ($f = 0, 1, 2, \ldots j$)

$$\sum_{t=0}^{t=j} \eta_{f,t} \epsilon_t = \sum_{t=0}^{t=l} \eta_{f,t} \delta_t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

(v.) The ordinary method of least squares would consist of making $\sum_s (v_s - u_s)^2$ a minimum, and would lead to equations

$$\sum_s (s_f)(v_s - u_s) = 0,$$

which would not give the most probable values of the ϵ's.

14. *Fitting by Moments.*—(i.) The ordinary method of moments, adapted to the case in which the δ's are not necessarily the successive differences of the u's, would consist in equating the values of $\sum_s (s_f) v_s$ and $\sum_s (s_f) u_s$. This, as will be seen from § 13 (v.), would not give the most probable values of the ϵ's.

(ii.) In order to obtain the most probable values of the ϵ's by equating moments of the v's and of the u's, we must write (say)

$$M_f \equiv \sum_s [s_f] u_s, \quad \ldots \quad (71)$$

and define the f^{th} moment of the u's as being M_f or a definite l.c. of M_f, M_{f-1}, $M_{f-2}, \ldots M_0$. But the coefficient of u_s in M_f would then not be given definitely by the relations between the u's and the δ's, but would depend also on the law of correlation of errors of the u's. We see, however, from § 13 (iii.), that we have also

$$M_f = \sum_r (r_f) y_r, \quad \ldots \quad (72)$$

and that we get the same result by equating moments, defined in this way, of the y's and the z's. In the ordinary case in which the δ's are successive differences of the u's, the coefficients of the y's in (72) are binomial coefficients, and the ordinary moments fall within the definition given above. It follows that *in fitting a polynomial to a set of quantities* (not being a self-conjugate set) *by the method of moments, the moments which ought to be equated are not those of the quantities themselves and their assumed values, but those of the conjugate set of the former and the corresponding l.cc. of the latter.*

15. *Reduction of Error.*—If we improve the δ's or the u's by means of the δ's after δ_j, the improved values of these latter are zero, and those of the δ's up to δ_j are obtainable from (XXI.) of § 8, which states that the improved values of the σ's from σ_0 to σ_j are equal to the original values. Using (6), this gives ($f = 0, 1, 2, \ldots j$)

$$\sum_{t=0}^{t=j} \eta_{f,t} (\epsilon_t)_j = \sum_{t=0}^{t=l} \eta_{f,t} \delta_t. \quad \ldots \quad (73)$$

Comparing this with (70), we see that the ϵ's given by this process are the same as those given by the process of fitting the expression in (57).

16. *Difference of the Two Processes.*—Although the two processes lead to the

same result, they are essentially different. This is explained in § 22 of "Reduction." The main difference may be expressed as follows:—

(i.) In "fitting" we deal directly with the particular case. We assume that the true values follow a specified law, involving unknown constants, and we deduce values for these constants from the data by the principle of inverse probability.

(ii.) In "reduction of error" we do not use inverse probability, and we only deal incidentally with the particular case. We regard the aggregate of the data as one of an indefinitely great number of possible aggregates from the same true values, and we use a method which will reduce as much as possible the m.s.e. of these possible aggregates.

Some Steps in the General Solution.

17. *Preliminary.*—(i.) Our object is to find the improved value of any l.c. of the δ's or the u's, and the m.s.e. of this improved value or the m.p.e. of two improved values. Ordinarily, as already stated in § 8 (iv.), the quantity to be improved would be expressible in terms of the δ's, so that we need consider only the improved values of the δ's, *i.e.*, the ϵ's. There are then four problems before us, viz.: (1) expression of the ϵ's in terms of the δ's; (2) expression in terms of the σ's; (3) expression in terms of the y's; (4) determination of the λ's. For practical purposes (2) is more important than (1) or (3), since there will be fewer coefficients involved.

(ii.) Although it does not seem possible to obtain a general solution, otherwise than by determinants, there are some general propositions that indicate stages in the solution. If, without necessarily finding the complete expressions of the ϵ's in terms of the δ's, we can find for each ϵ the coefficient of the first of the auxiliaries, then it will be seen from § 18 that we can find all the ϵ's if we know the E's, and from § 19 (i.) that we can find all the λ's if we know the Λ's. It follows from (40) that in this latter case we can at once obtain the ϵ's in terms of the σ's.

(iii.) We use the notation of § 7, and we also write

$$-\theta_{f,j} \equiv \text{coefficient of } \delta_j \text{ (as auxiliary) in } (\epsilon_f)_{j-1},$$

so that

$$\theta_{f,j} = 0 \text{ if } f > j, \quad \ldots \ldots \ldots \quad (74)$$

$$\theta_{j,j} = 1. \quad \ldots \ldots \ldots \quad (75)$$

It should be observed that $\theta_{f,j}$ is not equal to $\theta_{j,f}$. The θ's may be known directly, or, as is shown in (83), we may be able to obtain them from certain of the λ's.

18. *Formula of Progression.*—The quantities which we want to find are the improved values

of δ_0, using $\delta_1, \delta_2, \delta_3, \ldots$,

of δ_1 and δ_0, using $\delta_2, \delta_3, \delta_4, \ldots$,

of δ_2, δ_1, and δ_0, using $\delta_3, \delta_4, \delta_5, \ldots$, and so on. There is a formula connecting these, which makes it unnecessary to deal with more than the first quantity in each row; or, if we deal with them all, the formula can be used for checking the results. (An example is given at the end of § 15 of "Reduction.")

We have

$$(\epsilon_f)_{j-1} = \delta_f - \theta_{f,j}\delta_j + \text{ terms in } \delta_{j+1}, \delta_{j+2}, \ldots .$$

By (XV.), this is the improved value of $\delta_f - \theta_{f,j}\delta_j$, using δ's after δ_j; and therefore, by (XIII.),

$$(\epsilon_f)_{j-1} = (\epsilon_f)_j - \theta_{f,j}(\epsilon_j)_j = (\epsilon_f)_j - \theta_{f,j}\mathrm{E}_j. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

Re-arranging, and replacing j by $j-1$, $j-2$, ... $f+1$, and remembering that, by (75), $\theta_{f,f} = 1$, we have

$$\begin{aligned}
(\epsilon_f)_j \quad - (\epsilon_f)_{j-1} &= \theta_{f,j} \quad \mathrm{E}_j, \\
(\epsilon_f)_{j-1} - (\epsilon_f)_{j-2} &= \theta_{f,j-1}\mathrm{E}_{j-1}, \\
&\vdots \\
(\epsilon_f)_{f+1} - (\epsilon_f)_f \quad &= \theta_{f,f+1}\mathrm{E}_{f+1}, \\
(\epsilon_f)_f \qquad\qquad &= \theta_{f,f} \quad \mathrm{E}_f.
\end{aligned}$$

Hence, by addition,

$$(\epsilon_f)_j = \theta_{f,f}\mathrm{E}_f + \theta_{f,f+1}\mathrm{E}_{f+1} + \theta_{f,f+2}\mathrm{E}_{f+2} + \ldots + \theta_{f,j}\mathrm{E}_j. \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

19. *Mean Products of Error* (*Alternative Formula*).—(i.) By (77) and (38),

$$\begin{aligned}
(\lambda_{f,g})_j &= \text{m.p.e. of } \delta_g \text{ and } (\epsilon_f)_j \\
&= \text{m.p.e. of } \delta_g \text{ and } \theta_{f,f}\mathrm{E}_f + \theta_{f,f+1}\mathrm{E}_{f+1} + \ldots + \theta_{f,j}\mathrm{E}_j \\
&= \theta_{f,f}(\lambda_{g,f})_f + \theta_{f,f+1}(\lambda_{g,f+1})_{f+1} + \ldots + \theta_{f,j}(\lambda_{g,j})_j \\
&= \Sigma\theta_{f,t}(\lambda_{g,t})_t.
\end{aligned}$$

The summation has to be made from $t = f$ to $t = j$. But, if $g > f$, we see from (37) that it is sufficient to make the summation from $t = g$. Hence, using "$t = f, g$" to denote summation from $t = f$ or from $t = g$, according as f or g is the greater, we have

$$(\lambda_{f,g})_j = \sum_{t=f,\,g}^{t=j} \theta_{f,t}(\lambda_{g,t})_t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

But, by putting $g = j$ in (78) (or taking the m.p.e. of δ_j and each member of (76)) and then replacing f and j by g and t,

$$(\lambda_{g,t})_t = \theta_{g,t}\Lambda_t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79)$$

Hence, substituting in (78),

$$(\lambda_{f,g})_j = \sum_{t=f,\,g}^{t=j} \theta_{f,t}\theta_{g,t}\Lambda_t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (80)$$

If we can obtain the θ's and the Λ's in a simple form, we thus have a workable formula for calculating the λ's, and thence, by (40), for determining the ϵ's in terms of the σ's.

(ii.) From (80), using (II.) of § 6, we get the m.ss.e. and m.pp.e. of the improved values of any l.cc. of the δ's. Let

$$w \equiv b_0\delta_0 + b_1\delta_1 + \ldots + b_l\delta_l, \qquad w' \equiv c_0\delta_0 + c_1\delta_1 + \ldots + c_l\delta_l,$$

and let the improved values of w and w', using δ's after δ_j, be x and x'. Then

$$\text{m.p.e. of } x \text{ and } x' = \sum_{t=0}^{t=j} (b_0\theta_{0,t} + b_1\theta_{1,t} + \ldots)(c_0\theta_{0,t} + c_1\theta_{1,t} + \ldots)\Lambda_t$$

$$= \sum_{t=0}^{t=j} (b_0\theta_{0,t} + b_1\theta_{1,t} + \ldots + b_t\theta_{t,t})(c_0\theta_{0,t} + c_1\theta_{1,t} + \ldots + c_t\theta_{t,t})\Lambda_t, \quad (81)$$

$$\text{m.s.e. of } x = \sum_{t=0}^{t=j} (b_0\theta_{0,t} + b_1\theta_{1,t} + \ldots + b_t\theta_{t,t})^2\Lambda_t. \quad (82)$$

(iii.) We have assumed that the θ's are known. If they are not known directly, but the values of $(\lambda_{f,t})_t$ are known, then, by (79),

$$\theta_{f,t} = (\lambda_{f,t})_t/\Lambda_t. \quad (83)$$

Substituting in (80),

$$(\lambda_{f,g})_j = \sum_{t=f,g}^{t=j} (\lambda_{f,t})_t(\lambda_{g,t})_t/\Lambda_t. \quad (84)$$

Also (77) is replaced by

$$(\epsilon_f)_j = \sum_{t=f}^{t=j} (\lambda_{f,t})_t \mathrm{E}_t/\Lambda_t. \quad (85)$$

Application to Self-Conjugate Set.

20. *Preliminary.*—(i.) We have now to apply the preceding results to the case in which the u's are a self-conjugate set, so that $(u_r; u_s) = 0$ $(r \neq s)$, $(u_r; u_r) = 1$, $y_r = u_r$. We take the δ's to be successive differences of the u's, commencing with a difference of order 0. The δ's to be used as auxiliaries will be those following δ_j; the $(\quad)_j$ will usually be omitted. We shall take the number of u's or of δ's to be m, so that $m = l+1$.

(ii.) There will be three cases to be considered; advancing differences, and central differences for m odd and for m even. For advancing differences the u's will be taken to be $u_0, u_1, \ldots u_{m-1}$. For central differences we shall write $m = 2n+1$ or $m = 2n$; and the u's will be $u_{-n}, u_{-n+1}, \ldots u_n$ and $u_{-n+1}, u_{-n+2}, \ldots u_n$ respectively. We shall require the following m.pp.e., which can be obtained from ordinary difference formulæ.

$$\text{m.p.e. of } \Delta^f u_0 \text{ and } \Delta^g u_0 = (-)^{f-g}(f+g, f), \qquad (86)$$

$$\text{,,} \quad \delta^{2f} u_0 \quad \text{,,} \quad \delta^{2g} u_0 = (-)^{f-g}(2f+2g, f+g), \qquad (87)$$

$$\text{,,} \quad \delta^{2f} u_0 \quad \text{,,} \quad \mu\delta^{2g-1} u_0 = 0, \qquad (88)$$

$$\text{,,} \quad \mu\delta^{2f-1} u_0 \quad \text{,,} \quad \mu\delta^{2g-1} u_0 = (-)^{f-g}(2f+2g-2, f+g-1)/(2f+2g), \qquad (89)$$

$$\text{,,} \quad \delta^{2f-1} u_{\frac{1}{2}} \quad \text{,,} \quad \delta^{2g-1} u_{\frac{1}{2}} = (-)^{f-g}(2f+2g-2, f+g-1), \qquad (90)$$

$$\text{,,} \quad \delta^{2f-1} u_{\frac{1}{2}} \quad \text{,,} \quad \mu\delta^{2g-2} u_{\frac{1}{2}} = 0, \qquad (91)$$

$$\text{,,} \quad \mu\delta^{2f-2} u_{\frac{1}{2}} \quad \text{,,} \quad \mu\delta^{2g-2} u_{\frac{1}{2}} = (-)^{f-g}(2f+2g-4, f+g-2)/(2f+2g-2). \qquad (92)$$

(iii.) For advancing differences we shall have

$$\delta^f \equiv \Delta^f u_0, \qquad \epsilon_f \equiv \Delta^f v_0.$$

The formulæ will be marked (A).

(iv.) For central differences the two cases of m odd and m even must be considered separately; but it will be found that, when the formulæ relating to v_0, $\delta^2 v_0$, ... (m odd) and to $\delta v_{\frac{1}{2}}$, $\delta^3 v_{\frac{1}{2}}$, ... (m even) are properly expressed, they are practically identical in form, as also are those relating to $\mu\delta v_0$, $\mu\delta^3 v_0$ ... (m odd) and to $\mu v_{\frac{1}{2}}$, $\mu\delta^2 v_{\frac{1}{2}}$, ... ($m$ even); and the latter correspond to the former with certain interchanges of ( ] and [ ). We therefore, for $\mu\delta v_0$, $\mu\delta^3 v_0$, ... and $\mu v_{\frac{1}{2}}$, $\mu\delta^2 v_{\frac{1}{2}}$, ..., replace θ, E, Λ, λ, by ϕ, I, M, μ, with the appropriate suffixes.

(v.) For $m = 2n+1$ it will be seen from (88), taken with (XIV.) of § 6, that the differences of even and of odd order can be treated independently. The δ's will be u_0, $\delta^2 u_0$, ... $\delta^{2n} u_0$ in the one case and $\mu\delta u_0$, $\mu\delta^3 u_0$, ... $\mu\delta^{2n-1} u_0$ in the other. We shall denote these by δ_0, δ_2, ... δ_{2n} and δ_1, δ_3, ... δ_{2n-1} respectively, and shall take j to be $2k$ or $2k+1$ for the former and $2k-1$ or $2k$ for the latter. The subscripts of the θ's and the ϕ's will be modified accordingly; *i.e.*, $\theta_{2f, 2k}$ will mean the coefficient of $-\delta_{2k}$ in $(\epsilon_{2f})_{2k-2}$, and similarly for $\phi_{2f-1, 2k-1}$. The formulæ for the two cases will be marked (B) and (C) respectively.

(vi.) Similarly for $m = 2n$ we see from (91) that differences of odd and of even order can be treated separately. The δ's are δ_1, δ_3, ... δ_{2n-1} in the one case, and δ_0, δ_2, ... δ_{2n-2} in the other, where $\delta_{2f-1} \equiv \delta^{2f-1} u_{\frac{1}{2}}$, $\delta_{2f-2} \equiv \mu\delta^{2f-2} u_{\frac{1}{2}}$; and j is taken to be $2k-1$ or $2k$ for the former and $2k-2$ or $2k-1$ for the latter. Also $\theta_{2f-1, 2k-1}$ means the coefficient of $-\delta_{2k-1}$ in $(\epsilon_{2f-1})_{2k-3}$; and similarly for $\phi_{2f-2, 2k-2}$. The formulæ will be marked (D) and (E).

(vii.) Writing

$$F\{\alpha, \beta, \ldots ; \rho, \psi, \ldots\} \equiv 1+\frac{\alpha . \beta \ldots}{\rho . \psi \ldots}+\frac{\alpha(\alpha+1) . \beta(\beta+1) \ldots}{\rho(\rho+1) . \psi(\psi+1) \ldots}+\ldots,$$

where α is a negative integer, it should be noted that

$$F\{-n, \beta ; 1, \psi\}=\frac{[\psi-\beta, n]}{[\psi, n]}, \quad \ldots \ldots \quad (93)$$

and that, if

$$\alpha=-n, \quad \psi+\chi=-n+\beta+\gamma+1,$$

then*

$$F\{-n, \beta, \gamma ; 1, \psi, \chi\}=\frac{[\psi-\beta, n][\chi-\beta, n]}{[\psi, n][\chi, n]}=\frac{[\psi-\gamma, n][\chi-\gamma, n]}{[\psi, n][\chi, n]}. \quad (94)$$

(viii.) For the central-difference formulæ it will be convenient to write, if r and s are both even or both odd and $s \geqslant r$,

$$\{s, r\} \equiv (\tfrac{1}{2})^s \frac{(s+r, s)(s, \frac{1}{2}s+\frac{1}{2}r)}{r+1}, \quad \ldots \ldots \quad (95)$$

$$\{r, s\} \equiv (\tfrac{1}{2})^r \frac{(s+r, r)}{(r+1)(\frac{1}{2}r+\frac{1}{2}s, r)}; \quad \ldots \ldots \quad (96)$$

so that, if $k \geqslant f$,

$$\{2k+1, 2f+1\}=\frac{[f+\frac{3}{2}, k](k, f)}{2f+2}=\frac{[f+\frac{1}{2}, k+1](k+1, f+1)}{2f+1}, \quad (97)$$

$$\{2k, 2f\}=\frac{[f+\frac{1}{2}, k](k, f)}{2f+1}=\frac{[f+\frac{3}{2}, k-1](k-1, f-1)}{2f}, \quad (98)$$

and, if $k \leqslant s$,

$$\{2k+1, 2s+1\}=\frac{[s+\frac{3}{2}, k]}{(2k+2)(s, k)}=\frac{[s+\frac{1}{2}, k+1]}{(2s+1)(s, k)}, \quad \ldots \quad (99)$$

$$\{2k, 2s\}=\frac{[s+\frac{1}{2}, k]}{(2k+1)(s, k)}. \quad \ldots \ldots \quad (100)$$

(ix.) The successive steps are as follows. The formulæ for the ϵ's (the improved values of the differences) in terms of the differences have already been found in "Reduction" and "Fitting"; they depend on certain theorems as to the coefficients when l.cc. of moments or sums are expressed in terms of differences. From these formulæ we get the θ's, and also the E's; and thence we get, in each case, the progression formula supplied by (77). This formula is not really necessary, but it is useful for checking. The Λ's, *i.e.* the m.ss.e. of the E's, are found from (86)–(92), using (IV.) of § 6; this results in certain hypergeometric series, to which we apply (93) and (94). We then get expressions for the λ's, by (80). From these, by (40), we

* 'Proceedings of the London Mathematical Society,' 2nd series, x., 474.

have the values of the ϵ's in terms of the σ's; and also, by § 10 (iii.), the values in terms of the u's (which are identical with the y's). This completes the investigation of the improved values; but we also want to see the extent of the improvement. A further section therefore gives the ratio of the m.s.e. of the improved value to the m.s.e. of the original value, in a form convenient for calculation.

(x.) The m.p.e. of $\Delta^f u_r$ and $\Delta^f u_{r+t}$ is $(-)^t\,(2f, f+t)$. Hence, in finding the general solution of our problem for the case of a self-conjugate set, we are also finding it for the case of a set in which the m.p.e. of u_r and u_s is of the form $(-)^{sr}\,(2f, f+s-r)$, f being some positive integer; for we can treat these u's as the f^{th} differences of members of a self-conjugate set.

21. *Improved Values.*—(i.) Adapting § 11 (iii.) to the case in which the y's are identical with the u's, we see that, if w is any l.c. of the u's, its improved value, using differences of order exceeding j, is of the form $\Sigma p_r u_r$, where p_r is a polynomial of degree j in r. The improved values of the u's and their differences are obtained from this by means of certain formulæ, given in §§ 6 and 7 (iv.) of "Factorial Moments," for the expression of $\Sigma p_r u_r$ in terms of differences. The results are given in (12), (18), (19), (26), and (25) of "Fitting." From the first of these, replacing $m+1$ by m, we have $(f = 0, 1, 2, \ldots j)$

$$\text{(A)}\qquad \Delta^f v_0 = \sum_{s=0}^{s=m-1} (s, f)\,(j-s, j-f)\,\frac{(j+f+1, j)}{(j+s+1, j)}\,\frac{(m, s+1)}{(m, f+1)}\,\Delta^s u_0\,; \qquad (101)$$

and from the other four, altering k to $k-1$ in the last, we have $(f = 0, 1, 2, \ldots k$ in (102) and $1, 2, \ldots k$ in (103)–(105))

$$\text{(B)}\qquad \delta^{2f} v_0 = \sum_{s=0}^{s=n} (k+s, k+f)\,(k-s, k-f)\,\frac{(2k+2f+1, 2k)}{(2k+2s+1, 2k)}\,\frac{[n+\tfrac{1}{2}, 2s+1)}{[n+\tfrac{1}{2}, 2f+1)}\,\delta^{2s} u_0\,. \qquad (102)$$

$$= \delta^{2f} u_0 + (-)^{k-f} \sum_{s=k+1}^{s=n} \frac{s-k}{s-f}\,\frac{2f+2}{2k+2}\,\frac{\{2k+1, 2f+1\}}{\{2k+1, 2s+1\}}\,\frac{[\tfrac{1}{2}m, 2s+1)}{[\tfrac{1}{2}m, 2f+1)}\,\delta^{2s} u_0, \qquad (102\text{A})$$

$$\text{(C)}\quad \mu\delta^{2f-1} v_0 = \sum_{s=1}^{s=n} (k+s-1, k+f-1)\,(k-s, k-f)\,\frac{(2k+2f-1, 2k-1)}{(2k+2s-1, 2k-1)}\,\frac{(n+\tfrac{1}{2}, 2s]}{(n+\tfrac{1}{2}, 2f]}\,\mu\delta^{2s-1} u_0 \qquad (103)$$

$$= \mu\delta^{2f-1} u_0 + (-)^{k-f} \sum_{s=k+1}^{s=n} \frac{s-k}{s-f}\,\frac{2f+1}{2k+1}\,\frac{\{2k, 2f\}}{\{2k, 2s\}}\,\frac{(\tfrac{1}{2}m, 2s]}{(\tfrac{1}{2}m, 2f]}\,\mu\delta^{2s-1} u_0, \qquad (103\text{A})$$

$$\text{(D)}\quad \delta^{2f-1} v_{\frac{1}{2}} = \sum_{s=1}^{s=n} (k+s-1, k+f-1)\,(k-s, k-f)\,\frac{(2k+2f-1, 2k-1)}{(2k+2s-1, 2k-1)}\,\frac{[n, 2s)}{[n, 2f)}\,\delta^{2s-1} u_{\frac{1}{2}} \qquad (104)$$

$$= \delta^{2f-1} u_{\frac{1}{2}} + (-)^{k-f} \sum_{s=k+1}^{s=n} \frac{s-k}{s-f}\,\frac{2f+1}{2k+1}\,\frac{\{2k, 2f\}}{\{2k, 2s\}}\,\frac{[\tfrac{1}{2}m, 2s)}{[\tfrac{1}{2}m, 2f)}\,\delta^{2s-1} u_{\frac{1}{2}}, \qquad (104\text{A})$$

(E) $$\mu\delta^{2f-2}v_{\frac{1}{2}} = \sum_{s=1}^{s=n}(k+s-2,\ k+f-2)\,(k-s,\ k-f)\,\frac{(2k+2f-3,\ 2k-2)}{(2k+2s-3,\ 2k-2)}\,\frac{(n,\ 2s-1]}{(n,\ 2f-1]}\,\mu\delta^{2s-2}u_{\frac{1}{2}} \quad \ldots \quad (105)$$

$$= \mu\delta^{2f-2}u_{\frac{1}{2}} + (-)^{k-f}\sum_{s=k+1}^{s=n}\frac{s-k}{s-f}\,\frac{2f}{2k}\,\frac{\{2k-1,\ 2f-1\}}{\{2k-1,\ 2s-1\}}\,\frac{(\frac{1}{2}m,\ 2s-1]}{(\frac{1}{2}m,\ 2f-1]}\,\mu\delta^{2s-2}u_{\frac{1}{2}}. \quad \ldots \quad (105\text{A})$$

(ii.) From (i.) we obtain

(A) $$\theta_{f,j} = (-)^{j-f}(j,f)\,\frac{(j+f,\ j-1)}{(2j,\ j-1)}\,\frac{(m,\ j+1)}{(m,\ f+1)}, \quad \ldots \quad (106)$$

(B) $$\theta_{2f,2k} = (-)^{k-f}\,\frac{\{2k,\ 2f\}}{\{2k,\ 2k\}}\,\frac{[\frac{1}{2}m,\ 2k+1)}{[\frac{1}{2}m,\ 2f+1)}, \quad \ldots \quad (107)$$

(C) $$\phi_{2f-1,2k-1} = (-)^{k-f}\,\frac{\{2k-1,\ 2f-1\}}{\{2k-1,\ 2k-1\}}\,\frac{(\frac{1}{2}m,\ 2k]}{(\frac{1}{2}m,\ 2f]}, \quad \ldots \quad (108)$$

(D) $$\theta_{2f-1,2k-1} = (-)^{k-f}\,\frac{\{2k-1,\ 2f-1\}}{\{2k-1,\ 2k-1\}}\,\frac{[\frac{1}{2}m,\ 2k)}{[\frac{1}{2}m,\ 2f)}, \quad \ldots \quad (109)$$

(E) $$\phi_{2f-2,2k-2} = (-)^{k-f}\,\frac{\{2k-2,\ 2f-2\}}{\{2k-2,\ 2k-2\}}\,\frac{(\frac{1}{2}m,\ 2k-1]}{(\frac{1}{2}m,\ 2f-1]}. \quad \ldots \quad (110)$$

(iii.) Also, by putting $f = j$ in (101) and $f = k$ in (102)–(105),

(A) $$\text{E}_j = \Delta^j v_0 = \sum_{s=j}^{s=m-1}(s,\ j)\,\frac{(2j+1,\ j)}{(j+s+1,\ j)}\,\frac{(m,\ s+1)}{(m,\ j+1)}\,\Delta^s u_0, \quad \ldots \quad (111)$$

(B) $$\text{E}_{2k} = \delta^{2k}v_0 = \sum_{s=k}^{s=n}\frac{\{2k+1,\ 2k+1\}}{\{2k+1,\ 2s+1\}}\,\frac{[\frac{1}{2}m,\ 2s+1)}{[\frac{1}{2}m,\ 2k+1)}\,\delta^{2s}u_0, \quad \ldots \quad (112)$$

(C) $$\text{I}_{2k-1} = \mu\delta^{2k-1}v_0 = \sum_{s=k}^{s=n}\frac{\{2k,\ 2k\}}{\{2k,\ 2s\}}\,\frac{(\frac{1}{2}m,\ 2s]}{(\frac{1}{2}m,\ 2k]}\,\mu\delta^{2s-1}u_0, \quad \ldots \quad (113)$$

(D) $$\text{E}_{2k-1} = \delta^{2k-1}v_{\frac{1}{2}} = \sum_{s=k}^{s=n}\frac{\{2k,\ 2k\}}{\{2k,\ 2s\}}\,\frac{[\frac{1}{2}m,\ 2s)}{[\frac{1}{2}m,\ 2k)}\,\delta^{2s-1}u_{\frac{1}{2}}, \quad \ldots \quad (114)$$

(E) $$\text{I}_{2k-2} = \mu\delta^{2k-2}v_{\frac{1}{2}} = \sum_{s=k}^{s=n}\frac{\{2k-1,\ 2k-1\}}{\{2k-1,\ 2s-1\}}\,\frac{(\frac{1}{2}m,\ 2s-1]}{(\frac{1}{2}m,\ 2k-1]}\,\mu\delta^{2s-2}u_{\frac{1}{2}}. \quad \ldots \quad (115)$$

(iv.) It has been pointed out in § 11 (iv.) that the differences of order j all have the same improved value. It follows that (112)–(115) are particular cases of (111), expressed in terms of central differences; the proper values being taken for m and for j, and u_0 being altered to u_{-n} for (112) and (113) and to u_{-n+1} for (114) and (115). We can verify this by expressing the E's in terms of the differences of order j. For (A) we have

$$\Delta^s u_0 = \{(1+\Delta)-1\}^{s-j}\,\Delta^j u_0 = \Delta^j u_{s-j} - (s-j,\ 1)\,\Delta^j u_{s-j-1} + \ldots.$$

Substituting in (111), and rearranging the terms, it will be found that the coefficient of $\Delta^j u_r$ is

$$(j+r, j)\frac{(2j+1, j)(m, j+r+1)}{(2j+r+1, j)(m, j+1)} F\{-m+j+r+1, j+r+1 ; 1, 2j+r+2\}$$
$$= (j+r, j)(m-r-1, j)/(m, 2j+1];$$

so that

(A) $$\mathrm{E}_j = \sum_{r=0}^{r=m-j-1} (j+r, j)\frac{(m-r-1, j)}{(m, 2j+1]}\Delta^j u_r. \quad . \quad . \quad . \quad . \quad . \quad (116)$$

Similarly from (112)–(115)

(B) $$\mathrm{E}_{2k} = \sum_{r=-n+k}^{r=n-k} \frac{(n+k+r, 2k)(n+k-r, 2k)}{(m, 4k+1]}\delta^{2k}u_r, \quad . \quad . \quad . \quad (117)$$

(C) $$\mathrm{I}_{2k-1} = \sum_{r=0}^{r=n-k} \frac{(n+k+r, 2k-1)(n+k-r-1, 2k-1)}{(m, 4k-1]}(\delta^{2k-1}u_{-r-\frac{1}{2}}+\delta^{2k-1}u_{r+\frac{1}{2}}), \quad . \quad (118)$$

(D) $$\mathrm{E}_{2k-1} = \sum_{r=-n+k}^{r=n-k} \frac{(n+k+r-1, 2k-1)(n+k-r-1, 2k-1)}{(m, 4k-1]}\delta^{2k-1}u_{r+\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad (119)$$

(E) $$\mathrm{I}_{2k-2} = \sum_{r=0}^{r=n-k} \frac{(n+k+r-1, 2k-2)(n+k-r-2, 2k-2)}{(m, 4k-3]}(\delta^{2k-2}u_{-r}+\delta^{2k-2}u_{r+1}). \quad . \quad (120)$$

The identity of (117)–(120) with (116), the u's being altered as explained above, is easily verified.

(v.) The formula of progression (77) takes simple forms if we attach the factors involving m to the δ's; for m then disappears from the formula.

(A) Writing

$$A_t \equiv (m, t+1)\mathrm{E}_t = (m, t+1)(\Delta^t v_0)_t,$$

so that, in effect, we take δ_f to be $(m, f+1)\Delta^f u_0$, (77) gives

(A) $$(m, f+1)\Delta^f v_0 = \sum_{t=f}^{t=j} (-)^{t-f}(t, f)\frac{(t+f, t-1)}{(2t, t-1)}A_t. \quad . \quad . \quad . \quad . \quad (121)$$

For $j = 3$, for instance, we should have

$$\left.\begin{aligned}(m, 1)v_0 &= A_0 - A_1 + \tfrac{1}{2}A_2 - \tfrac{1}{5}A_3\\(m, 2)\Delta v_0 &= \qquad A_1 - \tfrac{3}{2}A_2 + \tfrac{6}{5}A_3\\(m, 3)\Delta^2 v_0 &= \qquad\qquad A_2 - 2A_3\\(m, 4)\Delta^3 v_0 &= \qquad\qquad\qquad A_3\end{aligned}\right\},$$

where A_0 is the value of $(m, 1)v_0$ for $j = 0$, A_1 is the value of $(m, 2)\Delta v_0$ for $j = 1$, and so on. This may be verified by § 5 (i.)–(iii.) of "Fitting."

(B) (C) Writing

$$P_{2t} \equiv [\tfrac{1}{2}m, 2t+1)\mathrm{E}_{2t}, \; Q_{2t-1} \equiv (\tfrac{1}{2}m, 2t]\mathrm{I}_{2t-1},$$

we have

(B) $$[\tfrac{1}{2}m, 2f+1)\,\delta^{2f}v_0 = \sum_{t=f}^{t=k}(-)^{t-f}\frac{\{2t, 2f\}}{\{2t, 2t\}}P_{2t}, \quad (122)$$

(C) $$(\tfrac{1}{2}m, 2f]\,\mu\delta^{2f-1}v_0 = \sum_{t=f}^{t=k}(-)^{t-f}\frac{\{2t-1, 2f-1\}}{\{2t-1, 2t-1\}}Q_{2t-1}. \quad (123)$$

The first of these has been given, for $f = 0$, in "Reduction," § 15 (v.), p. 362; the notation of P_{2t} differing, however, in a factor $\tfrac{1}{2}m$.

(D) (E) Writing

$$P_{2t-1} \equiv [\tfrac{1}{2}m, 2t)\,\mathrm{E}_{2t-1},\quad Q_{2t-2} \equiv (\tfrac{1}{2}m, 2t-1]\,\mathrm{I}_{2t-2},$$

we have

(D) $$[\tfrac{1}{2}m, 2f)\,\delta^{2f-1}v_{\frac{1}{2}} = \sum_{t=f}^{t=k}(-)^{t-f}\frac{\{2t-1, 2f-1\}}{\{2t-1, 2t-1\}}P_{2t-1}, \quad (124)$$

(E) $$(\tfrac{1}{2}m, 2f-1]\,\mu\delta^{2f-2}v_{\frac{1}{2}} = \sum_{t=f}^{t=k}(-)^{t-f}\frac{\{2t-2, 2f-2\}}{\{2t-2, 2t-2\}}Q_{2t-2}. \quad (125)$$

(vi.) The Λ's are obtained from the E's by means of (IV.) of § 6; *e.g.*, for advancing differences, m.s.e. of E_j = m.p.e. of E_j and $\Delta^j u_0$. We can use either the values of the E's given by (111)–(115) or those given by (116)–(120); for the former we require the m.pp.e. given in (86), (87), (89), (90), (92), and for the latter the value of the m.p.e. of two differences of order f as given in § 20 (x.). Using the former method, we get the following results:—

(A) $$\begin{aligned}\Lambda_j &= \sum_{s=j}^{s=m-1}(-)^{s-j}(s, j)\frac{(2j+1, j)}{(s+j+1, j)}\frac{(m, s+1)}{(m, j+1)}(s+j, j)\\ &= (2j, j)\,F\{-m+j+1, 2j+1;\ 1, 2j+2\}\\ &= (2j, j)\,(m-j-1)!\,(2j+1)!/(m+j)!\\ &= (2j, j)/(m, 2j+1]; \quad (126)\end{aligned}$$

(B) $$\begin{aligned}\Lambda_{2k} &= \sum_{s=k}^{s=n}(-)^{s-k}\frac{\{2k+1, 2k+1\}}{\{2k+1, 2s+1\}}\frac{[\frac{1}{2}m, 2s+1)}{[\frac{1}{2}m, 2k+1)}(2s+2k, s+k)\\ &= (4k, 2k)\,F\{-n+k, n+k+1, 2k+\tfrac{1}{2};\ 1, 2k+1, 2k+\tfrac{3}{2}\}\\ &= (4k, 2k)/(m, 4k+1]; \quad (127)\end{aligned}$$

(C) $$\begin{aligned}\mathrm{M}_{2k-1} &= \sum_{s=k}^{s=n}(-)^{s-k}\frac{\{2k, 2k\}}{\{2k, 2s\}}\frac{(\frac{1}{2}m, 2s]}{(\frac{1}{2}m, 2k]}\frac{(2s+2k-2, s+k-1)}{2s+2k}\\ &= (4k-2, 2k-1)/(4k)\,.\,F\{-n+k, n+k+1, 2k-\tfrac{1}{2};\ 1, 2k+1, 2k+\tfrac{1}{2}\}\\ &= (4k-2, 2k-1)/(m, 4k-1]; \quad (128)\end{aligned}$$

(D) $$\begin{aligned}\Lambda_{2k-1} &= \sum_{s=k}^{s=n}(-)^{s-k}\frac{\{2k, 2k\}}{\{2k, 2s\}}\frac{[\frac{1}{2}m, 2s)}{[\frac{1}{2}m, 2k)}(2s+2k-2, s+k-1)\\ &= (4k-2, 2k-1)\,F\{-n+k, n+k, 2k-\tfrac{1}{2};\ 1, 2k, 2k+\tfrac{1}{2}\}\\ &= (4k-2, 2k-1)/(m, 4k-1]; \quad (129)\end{aligned}$$

(E) $$\mathrm{M}_{2k-2} = \sum_{s=k}^{s=n} (-)^{s-k} \frac{\{2k-1,\ 2k-1\}}{\{2k-1,\ 2s-1\}} \frac{(\tfrac{1}{2}m,\ 2s-1]}{(\tfrac{1}{2}m,\ 2k-1]} \frac{(2s+2k-4,\ s+k-2)}{2s+2k-2}$$
$$= (4k-4,\ 2k-2)/(4k-2)\,.\,F\{-n+k,\ n+k,\ 2k-\tfrac{3}{2}\,;\ 1,\ 2k,\ 2k-\tfrac{1}{2}\}$$
$$= (4k-4,\ 2k-2)/(m,\ 4k-3]. \qquad (130)$$

It would, of course, in view of the identity of the E's as stated in (iv.) above, have been sufficient to obtain (126) and deduce (127)–(130) from it.

(vii.) The Λ's having been found, the λ's, *i.e.*, the m.pp.e. of the improved values of $\Delta^f u_0$ and $\Delta^g u_0$, etc., are obtained by (80). The θ's and φ's being as in (ii.) above,

(A) $$\lambda_{f,g} = \sum_{t=f,g}^{t=j} \theta_{f,t}\,\theta_{g,t}\,(2t,\ t)/(m,\ 2t+1], \qquad (131)$$

(B) $$\lambda_{2f,2g} = \sum_{t=f,g}^{t=k} \theta_{2f,2t}\,\theta_{2g,2t}\,(4t,\ 2t)/(m,\ 4t+1], \qquad (132)$$

(C) $$\mu_{2f-1,2g-1} = \sum_{t=f,g}^{t=k} \phi_{2f-1,2t-1}\,\phi_{2g-1,2t-1}\,(4t-2,\ 2t-1)/(m,\ 4t-1], \qquad (133)$$

(D) $$\lambda_{2f-1,2g-1} = \sum_{t=f,g}^{t=k} \theta_{2f-1,2t-1}\,\theta_{2g-1,2t-1}\,(4t-2,\ 2t-1)/(m,\ 4t-1], \qquad (134)$$

(E) $$\mu_{2f-2,2g-2} = \sum_{t=f,g}^{t=k} \phi_{2f-2,2t-2}\,\phi_{2g-2,2t-2}\,(4t-4,\ 2t-2)/(m,\ 4t-3]. \qquad (135)$$

To find the m.s.e. of the improved value of any l.c. of the differences, or the m.p.e. of two such l.cc., we apply (II.) of § 6, as in § 19 (ii.). Thus, for $m = 2n+1$,

m.p.e. of $b_0v_0+b_1\mu\delta v_0+b_2\delta^2 v_0+\ldots+b_{2k}\delta^{2k}v_0$ and $c_0v_0+c_1\mu\delta v_0+c_2\delta^2 v_0+\ldots+c_{2k}\delta^{2k}v_0$

$$= \sum_{t=0}^{t=k}\left\{\sum_{f=0}^{f=t} b_{2f}\,\theta_{2f,2t}\right\}\left\{\sum_{g=0}^{g=t} c_{2g}\theta_{2g,2t}\right\}(4t,\ 2t)/(m,\ 4t+1]$$
$$+\sum_{t=1}^{t=k}\left\{\sum_{f=1}^{f=t} b_{2f-1}\phi_{2f-1,2t-1}\right\}\left\{\sum_{g=1}^{g=t} c_{2g-1}\phi_{2g-1,2t-1}\right\}(4t-2,\ 2t-1)/(m,\ 4t-1].$$

(viii.) The λ's having been found, the ε's are then given in terms of the σ's by (40). Using the expressions for the σ's given in (26) and (31)–(35), we get the following :—

(A) $$\Delta^f v_0 = \sum_{g=0}^{g=j} \lambda_{f,g}\Sigma''^{g+1}u_g \qquad (136)$$

$$= \left[L\left\{\sum_{g=0}^{g=j}(-)^g\lambda_{f,g}\,\Sigma^{g+1}u_g\,;\ \Sigma u_t\right\}\right]_{t=0}^{t=m}; \qquad (136\text{A})$$

(B) $$\delta^{2f}v_0 = \left[L\left\{\sum_{g=0}^{g=k}\lambda_{2f,2g}\,\mu\sigma^{2g+1}u_0\,;\ \sigma u_t\right\}\right]_{t=-\frac{1}{2}m}^{t=\frac{1}{2}m}; \qquad (137)$$

(C) $$\mu\delta^{2f-1}v_0 = \left[L\left\{-\sum_{g=1}^{g=k}\mu_{2f-1,2g-1}\sigma^{2g}u_0\,;\ \sigma u_t\right\}\right]_{t=-\frac{1}{2}m}^{t=\frac{1}{2}m}; \qquad (138)$$

(D) $$\delta^{2f-1}v_{\frac{1}{2}} = \left[L\left\{-\sum_{g=1}^{g=k}\lambda_{2f-1,2g-1}\mu\sigma^{2g}u_{\frac{1}{2}};\ \sigma u_{t+\frac{1}{2}}\right\}\right]_{t=-\frac{1}{2}m}^{t=\frac{1}{2}m}; \quad (139)$$

(E) $$\mu\delta^{2f-2}v_{\frac{1}{2}} = \left[L\left\{\sum_{g=1}^{g=k}\mu_{2f-2,2g-2}\sigma^{2g-1}u_{\frac{1}{2}};\ \sigma u_{t+\frac{1}{2}}\right\}\right]_{t=-\frac{1}{2}m}^{t=\frac{1}{2}m}. \quad (140)$$

The expression in (136) differs slightly from that given for $\Delta^f v_0$ in "Fitting," (15) and § 5; see Appendix III.

(ix.) Finally, we want to express the ϵ's in terms of the u's. This is done by means of the general theorem in § 10 (iii.), that the coefficients of the y's in any ϵ are related to the m.pp.e. of this ϵ and the ϵ's in the same way that the v's are related to the ϵ's. Thus we find that—

(A) If $\Delta^f v_0 = \sum_{r=0}^{r=m} p_r u_r$, then $p_r = \sum_{g=0}^{g=j}(r, g)\lambda_{f,g}$; (141)

(B) If $\delta^{2f} v_0 = \sum_{r=-n}^{r=n} p_r u_r$, then $p_r = \sum_{g=0}^{g=k}[r, 2g)\lambda_{2f,2g}$; (142)

(C) If $\mu\delta^{2f-1} v_0 = \sum_{r=-n}^{r=n} p_r u_r$, then $p_r = \sum_{g=1}^{g=k}(r, 2g-1]\mu_{2f-1,2g-1}$; (143)

(D) If $\delta^{2f-1} v_{\frac{1}{2}} = \sum_{r=-n+1}^{r=n} p_r u_r$, then $p_r = \sum_{g=1}^{g=k}[r-\frac{1}{2}, 2g-1)\lambda_{2f-1,2g-1}$; (144)

(E) If $\mu\delta^{2f-2} v_{\frac{1}{2}} = \sum_{r=-n+1}^{r=n} p_r u_r$, then $p_r = \sum_{g=1}^{g=k}(r-\frac{1}{2}, 2g-2]\mu_{2f-2,2g-2}$. (145)

For a comparison of these formulæ with those given in "Fitting," see Appendix IV.

22. *Extent of Improvement* (*Central Differences*).—A question of practical importance is the extent to which the use of these formulæ actually reduces the m.s.e. of some selected quantity, such as, for the cases marked (B), u_0 or $\delta^{2f}u_0$. The m.ss.e. of the various improved values are found from (131)–(135), by putting $g = f$. Comparing these with the m.ss.e. of the original values, for the central-difference formulæ (which are the important ones for practical use), we obtain the following:—

(B) $$\frac{\text{m.s.e. of } \delta^{2f}v_0}{\text{m.s.e. of } \delta^{2f}u_0} = \frac{1}{(4f,\ 2f)}\sum_{t=f}^{t=k}\left\{\frac{\{2t,\ 2f\}}{\{2t,\ 2t\}}\frac{[\frac{1}{2}m,\ 2t+1)}{[\frac{1}{2}m,\ 2f+1)}\right\}^2\frac{(4t,\ 2t)}{(m,\ 4t+1]}$$

$$= \frac{1}{(m,\ 4f+1]}\sum_{t=f}^{t=k}\frac{4t+1}{4f+1}\left\{\frac{\{2t,\ 2f\}}{\{2f,\ 2f\}}\right\}^2$$

$$\frac{\{m^2-(2f+1)^2\}\{m^2-(2f+3)^2\}\ldots\{m^2-(2t-1)^2\}}{\{m^2-(2f+2)^2\}\{m^2-(2f+4)^2\}\ldots\{m^2-(2t)^2\}}; \quad (146)$$

$$\text{(C)}\quad \frac{\text{m.s.e. of } \mu\delta^{2f-1}v_0}{\text{m.s.e. of } \mu\delta^{2f-1}u_0} = \frac{4f}{(4f-2,\ 2f-1)} \sum_{t=f}^{t=k} \left\{ \frac{\{2t-1, 2f-1\}}{\{2t-1,\ 2t-1\}} \frac{(\tfrac{1}{2}m,\ 2t]}{(\tfrac{1}{2}m, 2f]} \right\}^2 \frac{(4t-2,\ 2t-1)}{(m,\ 4t-1]}$$

$$= \frac{4f}{(m,\ 4f-1]} \sum_{t=f}^{t=k} \frac{4t-1}{4f-1} \left\{ \frac{\{2t-1,\ 2f-1\}}{\{2f-1, 2f-1\}} \right\}^2$$

$$\frac{\{m^2-(2f+1)^2\}\ \{m^2-(2f+3)^2\}\ \dots\ \{m^2-(2t-1)^2\}}{\{m^2-(2f)^2\}\ \{m^2-(2f+2)^2\}\ \dots\ \{m^2-(2t-2)^2\}}; \quad (147)$$

$$\text{(D)}\quad \frac{\text{m.s.e. of } \delta^{2f-1}v_{\frac{1}{2}}}{\text{m.s.e. of } \delta^{2f-1}u_{\frac{1}{2}}} = \frac{1}{(4f-2,\ 2f-1)} \sum_{t=f}^{t=k} \left\{ \frac{\{2t-1, 2f-1\}}{\{2t-1,\ 2t-1\}} \frac{[\tfrac{1}{2}m,\ 2t)}{[\tfrac{1}{2}m, 2f)} \right\}^2 \frac{(4t-2,\ 2t-1)}{(m,\ 4t-1]}$$

$$= \frac{1}{(m,\ 4f-1]} \sum_{t=f}^{t=k} \frac{4t-1}{4f-1} \left\{ \frac{\{2t-1,\ 2f-1\}}{\{2f-1, 2f-1\}} \right\}^2$$

$$\frac{\{m^2-(2f)^2\}\ \{m^2-(2f+2)^2\}\ \dots\ \{m^2-(2t-2)^2\}}{\{m^2-(2f+1)^2\}\ \{m^2-(2f+3)^2\}\ \dots\ \{m^2-(2t-1)^2\}}; \quad (148)$$

$$\text{(E)}\quad \frac{\text{m.s.e. of } \mu\delta^{2f-2}v_{\frac{1}{2}}}{\text{m.s.e. of } \mu\delta^{2f-2}u_{\frac{1}{2}}} = \frac{4f-2}{(4f-4,\ 2f-2)} \sum_{t=f}^{t=k} \left\{ \frac{\{2t-2,\ 2f-2\}}{\{2t-2,\ 2t-2\}} \frac{(\tfrac{1}{2}m,\ 2t-1]}{(\tfrac{1}{2}m,\ 2f-1]} \right\}^2 \frac{(4t-4,\ 2t-2)}{(m,\ 4t-3]}$$

$$= \frac{4f-2}{(m,\ 4f-3]} \sum_{t=f}^{t=k} \frac{4t-3}{4f-3} \left\{ \frac{\{2t-2,\ 2f-2\}}{\{2f-2,\ 2f-2\}} \right\}^2$$

$$\frac{\{m^2-(2f)^2\}\ \{m^2-(2f+2)^2\}\ \dots\ \{m^2-(2t-2)^2\}}{\{m^2-(2f-1)^2\}\ \{m^2-(2f+1)^2\}\ \dots\ \{m^2-(2t-3)^2\}}. \quad (149)$$

23. *Smoothing.*—When we have a table containing a very large number of u's, a common method of procedure is to use a formula involving a limited number of terms and to apply it to successive sets of the u's for the purpose of obtaining a table to be substituted for the original table. Thus we might use a formula involving $2n+1$ terms, and apply it to $u_0, u_1, u_2, \dots u_{2n}$ for finding a new value w_n, then to $u_1, u_2, u_3, \dots u_{2n+1}$ for finding a new value w_{n+1}, and so on. These values having been obtained, a differenced table would be formed; but, as by hypothesis the true differences of order exceeding j are negligible, the table would only go up to differences of order j. There are two cases to be considered.

(i.) If our object is to obtain as accurate values as possible for the w's, consistently with our using only the specified number of u's for each, the most accurate values would be the v's given by the formulæ considered in this and the preceding papers. It should, however, be observed that the differences of the w's are not the same as the $\Delta^f v_0$, $\delta^{2f}v_0$, etc., occurring in those formulæ. Suppose, for instance, that we replace u_0 by its improved value v_0 obtained by means of the (B) formula involving $u_{-n}, u_{-n+1}, \dots u_n$, and replace u_1 by the improved value v_1 obtained in a similar way. The resulting value of $v_1 - v_0$ will involve the $2n+2$ u's from u_{-n} to u_{n+1}; but it will not be the same thing as the improved value $v_{\frac{1}{2}}$ obtained by the (D) formula involving these u's, and its m.s.e. will therefore be greater than that of the latter.

(ii.) If our object is to obtain a smooth table of the w's as a whole, we could do this by obtaining as accurate values as possible for the differences of the w's of order j.

The formula which would have to be applied to the u's in order to obtain this result can be constructed without difficulty. The important thing to notice is that, if we alter the differences of the u's and then obtain the w's from the altered differences by summation, the resulting values must be such as can be legitimately substituted for the u's. Suppose, for instance, that $j = 2k+1$, and that we use $2n+1$ u's for each w. The formula for w will have to be of the form

$$w_0 = u_0 + c_{2k+2}\delta^{2k+2}u_0 + c_{2k+4}\delta^{2k+4}u_0 + \ldots + c_{2n}\delta^{2n}u_0\,;$$

and this will give

$$\delta^{2k+1}w_{\frac{1}{2}} = \delta^{2k+1}u_{\frac{1}{2}} + c_{2k+2}\delta^{4k+3}u_{\frac{1}{2}} + c_{2k+4}\delta^{4k+5}u_{\frac{1}{2}} + \ldots + c_{2n}\delta^{2n+2k+1}u_{\frac{1}{2}}.$$

The problem of determining the c's so that the m.s.e. of $\delta^{2k+1}w_{\frac{1}{2}}$ shall be a minimum is the same as that of determining the coefficients in the improved value of $\delta^{2k+1}u_{\frac{1}{2}}$ for $j = 4k+1$ or $4k+2$, m being $2n+2k+2$; and the solution of this problem is given in § 21. Thus, in terms of sums, (139) gives

$$\delta^{2k+1}w_{\frac{1}{2}} = \left[L\left\{-\sum_{g=1}^{g=2k+1}\lambda_{2k+1,\,2g-1}\mu\sigma^{2g}u_{\frac{1}{2}}\;;\;\sigma u_{t+\frac{1}{2}}\right\}\right]_{t=-(n+k+1)}^{t=n+k+1}.$$

The λ's having been found, we shall then have, by summation,

$$w_0 = \left[L\left\{-\sum_{g=1}^{g=2k+1}\lambda_{2k+1,\,2g-1}\mu\sigma^{2g+2k+1}u_0\;;\;\sigma^{2k+2}u_t\right\}\right]_{t=-(n+k+1)}^{t=n+k+1}.$$

The ratio of the m.s.e. of $\delta^{2k+1}w_{\frac{1}{2}}$ to that of $\delta^{2k+1}u_{\frac{1}{2}}$ is given by (148).

Appendix I.—The Correlation-Determinant.

1. The m.p.e. of A and B being denoted by $(A\,;B)$, let

$$\Theta \equiv \begin{vmatrix} (A\,;A) & (A\,;B) & (A\,;C) & \ldots \\ (B\,;A) & (B\,;B) & (B\,;C) & \ldots \\ (C\,;A) & (C\,;B) & (C\,;C) & \ldots \\ \vdots & \vdots & \vdots & \end{vmatrix}$$

We call this the *correlation-determinant* of A, B, C, ...

2. The elements of this determinant may be regarded as obtained as follows. We first take a representative collection of N_A values of the error of A; N_A being usually indefinitely great. Then, for each of these values, we take a representative collection of N_B values of the error of B; the resulting N_A collections will all be alike if the errors of A and of B are independent, but not if they are correlated. This gives N_AN_B combinations of an error of A and an error of B. For each of these we take

a representative collection of N_C values of the error of C; and so on. Thus finally we shall have $N \equiv N_A N_B N_C \ldots$ combinations of an error of A, an error of B, &c. Numbering these $1, 2, \ldots N$, and denoting the errors of A, of B, of $C, \ldots$ by $a, b, c, \ldots$, the combinations will be $a_1, b_1, c_1, \ldots, a_2, b_2, c_2, \ldots, \ldots a_N, b_N, c_N, \ldots$; and we shall have

$$(A\,;A) = (a_1^2+a_2^2+a_3^2+\ldots+a_N^2)/N,$$
$$(A\,;B) = (a_1b_1+a_2b_2+a_3b_3+\ldots+a_Nb_N)/N,$$
$$\&c.$$

3. Substituting these values in Θ, we find that, if there are m of the quantities $A, B, C, \ldots$,

$$N^m\Theta = \begin{vmatrix} a_1^2+a_2^2+a_3^2+\ldots & a_1b_1+a_2b_2+a_3b_3+\ldots & a_1c_1+a_2c_2+a_3c_3+\ldots & \&c. \\ a_1b_1+a_2b_2+a_3b_3+\ldots & b_1^2+b_2^2+b_3^2+\ldots & b_1c_1+b_2c_2+b_3c_3+\ldots & \&c. \\ a_1c_1+a_2c_2+a_3c_3+\ldots & b_1c_1+b_2c_2+b_3c_3+\ldots & c_1^2+c_2^2+c_3^2+\ldots & \&c. \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

$$= (a_1b_2c_3\ldots)^2+(a_1b_2c_4\ldots)^2+(a_1b_3c_4\ldots)^2+(a_2b_3c_4\ldots)^2+\ldots,$$

where $(a_1b_2c_3\ldots)$ denotes $\begin{vmatrix} a_1a_2a_3\ldots \\ b_1b_2b_3\ldots \\ c_1c_2c_3\ldots \\ \vdots \end{vmatrix}$. Hence Θ is not $= 0$ unless each of the determinants $(abc\ldots)$ is $= 0$. This would be the case, for instance, if A were a constant, so that every a would be 0, or if there were a linear relation connecting the errors of $A, B, C, \ldots$.

4. Let Φ be the correlation-determinant of $A, B, \ldots P, Q, \ldots$, and Ψ that of $A, B, \ldots P$.

(a) Suppose that $\Psi = 0$. Then, by § 3 of this Appendix, each of the determinants $(ab \ldots p)$, where $a, b, \ldots p$ are the errors of $A, B, \ldots P$, is 0. But these are the minors of the q's in the determinants $(ab \ldots pq)$; and therefore these latter determinants are 0. Proceeding in this way, we see that the determinants $(ab\ldots pq\ldots)$ are all 0; and therefore $\Phi = 0$.

(b) Hence, if Φ is not $= 0$, Ψ is not $= 0$.

Appendix II.—Frequency of Correlated Errors.

1. Let $u_0, u_1, u_2, \ldots u_l$ and $y_0, y_1, y_2, \ldots y_l$ be two conjugate sets. Denote the errors of the u's by $\theta_0, \theta_1, \theta_2, \ldots \theta_l$; and let the resulting errors of the y's be $\phi_0, \phi_1, \phi_2, \ldots \phi_l$.

Then, on the assumption of normal correlation of errors, the frequency of joint occurrence of these θ's is proportional to

$$\exp - \tfrac{1}{2}P,$$

where P is a homogeneous quadratic function of the θ's. We want to prove that

(i.) the ϕ's are the partial differential coefficients of $\frac{1}{2}P$ with regard to the θ's, and conversely;

(ii.) $P = \theta_0\phi_0 + \theta_1\phi_1 + \theta_2\phi_2 + \ldots + \theta_l\phi_l$;

(iii.) $P = \psi_{0,0}\theta_0^{\,2} + 2\psi_{0,1}\theta_0\theta_1 + \psi_{1,1}\theta_1^{\,2} + \ldots + \psi_{l,l}\theta_l^2$, where $\psi_{f,g}$ is the m.p.e. of y_f and y_g; and similarly

(iv.) $P = \pi_{0,0}\phi_0^{\,2} + 2\pi_{0,1}\phi_0\phi_1 + \pi_{1,1}\phi_1^{\,2} + \ldots + \pi_{l,l}\phi_l^2$, where $\pi_{f,g}$ is the m.p.e of u_f and u_g.

2. Suppose that

$$P = a_{0,0}\theta_0^{\,2} + 2a_{0,1}\theta_0\theta_1 + a_{1,1}\theta_1^{\,2} + \ldots + a_{l,l}\theta_l^{\,2};$$

and let us, without making any assumption of conjugacy, write ($f = 0, 1, 2, \ldots l$)

$$\begin{aligned} y_f &\equiv a_{f,0}u_0 + a_{f,1}u_1 + a_{f,2}u_2 + \ldots + a_{f,l}u_l, \\ \phi_f &\equiv \text{error of } y_f \\ &= a_{f,0}\theta_0 + a_{f,1}\theta_1 + a_{f,2}\theta_2 + \ldots + a_{f,l}\theta_l \\ &= \tfrac{1}{2}dP/d\theta_f. \end{aligned}$$

Then, writing the subscripts in the order f, 0, 1, 2, ... l,

$$\begin{aligned} P &= a_{f,f}\theta_f^2 + 2a_{f,0}\theta_f\theta_0 + a_{0,0}\theta_0^{\,2} + \ldots + a_{l,l}\theta_l^{\,2} \\ &= \phi_f^2/a_{f,f} + Q, \end{aligned}$$

where Q does not contain θ_f.

3. The mean value of $\phi_f\theta_g$ is N_g/D, where

$$N_g \equiv \iiint \ldots \int \phi_f\theta_g \exp - \tfrac{1}{2}P \,.\, d\theta_f\, d\theta_0\, d\theta_1 \ldots d\theta_l,$$

$$D \equiv \iiint \ldots \int \exp - \tfrac{1}{2}P \,.\, d\theta_f\, d\theta_0\, d\theta_1 \ldots d\theta_l,$$

the integration being in each case from $-\infty$ to ∞. If we write

$$\psi \equiv \phi_f/\sqrt{a_{f,f}},$$

then

$$N_g = \iiint \ldots \int \psi\theta_g \exp - \tfrac{1}{2}\psi^2 \,.\, \exp - \tfrac{1}{2}Q \,.\, d\psi\, d\theta_0\, d\theta_1 \ldots d\theta_l.$$

(a) First, suppose that g is not $= f$. Then, integrating with regard to ψ,

$$N_g = 0.$$

(*b*) Next, suppose that $g = f$. Then

$$N_f = 1/a_{f,f} \,.\, \iiint \dots \int \phi_f (\phi_f - a_{f,0}\theta_0 - a_{f,1}\theta_1 - \dots - a_{f,l}\theta_l) \exp - \tfrac{1}{2}P \,.\, d\theta_f \, d\theta_0 \, d\theta_1 \dots d\theta_l$$

$$= 1/a_{f,f} \,.\, \iiint \dots \int \phi_f^2 \exp - \tfrac{1}{2}P \,.\, d\theta_f \, d\theta_0 \, d\theta_1 \dots d\theta_l,$$

by (*a*). Hence

$$N_f = 1/\sqrt{a_{f,f}} \,.\, \iiint \dots \int \psi^2 \exp - \tfrac{1}{2}\psi^2 \,.\, \exp - \tfrac{1}{2}Q \,.\, d\psi \, d\theta_0 \, d\theta_1 \dots d\theta_l$$

$$= \sqrt{(2\pi/a_{f,f})} \,.\, \iint \dots \int \exp - \tfrac{1}{2}Q \,.\, d\theta_0 \, d\theta_1 \dots d\theta_l.$$

Also

$$D = 1/\sqrt{a_{f,f}} \,.\, \iiint \dots \int \exp - \tfrac{1}{2}\psi^2 \,.\, \exp - \tfrac{1}{2}Q \,.\, d\psi \, d\theta_0 \, d\theta_1 \dots d\theta_l$$

$$= \sqrt{(2\pi/a_{f,f})} \,.\, \iint \exp - \tfrac{1}{2}Q \,.\, d\theta_0 \, d\theta_1 \dots d\theta_l.$$

Hence

$$N_f/D = 1.$$

4. Hence the y's are related to the u's in such a way that

$$\text{m.p.e. of } y_f \text{ and } u_g = 0 \; (g \neq f) \text{ or } 1 \; (g = f);$$

and therefore the u's and the y's are conjugate sets; which proves (i.). It follows that

$$a_{f,g} = \text{m.p.e. of } y_f \text{ and } y_g.$$

This proves (iii.); and (iv.) is the similar result which we should have obtained by expressing P in terms of the ϕ's. Also

$$\begin{aligned} P &= a_{0,0}\theta_0^2 + 2a_{0,1}\theta_0\theta_1 + a_{1,1}\theta_1^2 + \dots + a_{l,l}\theta_l^2 \\ &= \theta_0 (a_{0,0}\theta_0 + a_{0,1}\theta_1 + a_{0,2}\theta_2 + \dots + a_{0,l}\theta_l) \\ &\quad + \theta_1 (a_{1,0}\theta_0 + a_{1,1}\theta_1 + a_{1,2}\theta_2 + \dots + a_{1,l}\theta_l) \\ &\quad + \dots \\ &\quad + \theta_l (a_{l,0}\theta_0 + a_{l,1}\theta_1 + a_{l,2}\theta_2 + \dots + a_{l,l}\theta_l) \\ &= \theta_0\phi_0 + \theta_1\phi_1 + \theta_2\phi_2 + \dots + \theta_l\phi_l; \end{aligned}$$

which proves (ii.).

Appendix III.—Improved Advancing Differences in terms of Sums.

The expression for $\Delta^f v_0$ given by (136) differs from that given in (15) and § 5 of "Fitting," in that it involves $\Sigma'' u_0$, $\Sigma''^2 u_1$, $\Sigma''^3 u_2$, ..., instead of $\Sigma'' u_0$, $\Sigma''^2 u_0$, $\Sigma''^3 u_0$, ...

The new expressions are more convenient for calculation and for tabulation, since the coefficients are rather smaller and are symmetrically placed about a diagonal. For $j = 2$, $m = 13$, for instance, the formulæ given by "Fitting," § 5 (ii.), are

$$1001v_0 = +693S_1 - 198S_2 + 22S_3,$$
$$1001\Delta v_0 = -231S_1 + 88S_2 - 11S_3,$$
$$1001\Delta^2 v_0 = +35S_1 - 15S_2 + 2S_3,$$

where $S_1 \equiv \Sigma'' u_0$, $S_2 \equiv \Sigma''^2 u_0$, $S_3 \equiv \Sigma''^3 u_0$. If we write $\Sigma_1 \equiv \Sigma'' u_0$, $\Sigma_2 \equiv \Sigma''^2 u_1$, $\Sigma_3 \equiv \Sigma''^3 u_2$, these become (by (136), or by writing $S_1 = \Sigma_1$, $S_2 = \Sigma_1 + \Sigma_2$, $S_3 = \Sigma_1 + 2\Sigma_2 + \Sigma_3$)

$$1001v_0 = +517\Sigma_1 - 154\Sigma_2 + 22\Sigma_3,$$
$$1001\Delta v_0 = -154\Sigma_1 + 66\Sigma_2 - 11\Sigma_3,$$
$$1001\Delta^2 v_0 = +22\Sigma_1 - 11\Sigma_2 + 2\Sigma_3.$$

The symmetry of the coefficients is due to the fact that

$$\text{co. } \Sigma''^{g+1} u_g \text{ in } \Delta^f v_0 = \lambda_{f,g} = \lambda_{g,f} = \text{co. } \Sigma''^{f+1} u_f \text{ in } \Delta^g v_0.$$

For any particular value of m there will be only $\frac{1}{2}(j+1)(j+2)$ coefficients to be tabulated, instead of $(j+1)^2$.

Appendix IV.—Formulæ in terms of u's.

(i.) Formulæ for $\Delta^f v_0$, &c., in terms of the u's have already been given in (15a), (21), (22), (29), and (28) of "Fitting"; and the results in (141)–(145) of the present paper can be checked by comparing the different expressions for the coefficients of the u's. We should require to use the following identities :—

$$(r+h,\ h) = (r,\ 0)\ (h,\ 0) + (r,\ 1)\ (h,\ 1) + (r,\ 2)\ (h,\ 2) + \ldots,$$
$$(r,\ 2h] = (0,\ 2h] + [r,\ 2)\ (0,\ 2h-2] + [r,\ 4)\ (0,\ 2h-4] + \ldots,$$
$$[r,\ 2h-1) = (r,\ 1]\ [\pm\tfrac{1}{2},\ 2h-2) + (r,\ 3]\ [\pm\tfrac{1}{2},\ 2h-4) + (r,\ 5]\ [\pm\tfrac{1}{2},\ 2h-6) + \ldots,$$
$$(r-\tfrac{1}{2},\ 2h-1] = [r-\tfrac{1}{2},\ 1)\ (0,\ 2h-2] + [r-\tfrac{1}{2},\ 3)\ (0,\ 2h-4] + [r-\tfrac{1}{2},\ 5)\ (0,\ 2h-6] + \ldots,$$
$$[r-\tfrac{1}{2},\ 2h-2) = [\pm\tfrac{1}{2},\ 2h-2) + (r-\tfrac{1}{2},\ 2]\ [\pm\tfrac{1}{2},\ 2h-4) + (r-\tfrac{1}{2},\ 4]\ [\pm\tfrac{1}{2},\ 2h-6) + \ldots.$$

(ii.) Taking, for instance, the formula for $\delta^{2f} v_0$ when $m = 2n+1$, (21) of "Fitting" gives (replacing t by f)

$$(p_r)_{2k} = (-)^f \tfrac{1}{2} m \frac{\{2k+1,\ 2f+1\}}{[\tfrac{1}{2}m,\ 2f+1)} \sum_{h=0}^{h=k} (-)^h \frac{(f+1)(h+1)}{f+h+\tfrac{1}{2}} \frac{\{2k+1,\ 2h+1\}\ (r,\ 2h]}{(\tfrac{1}{2}m,\ 2h+1]}.$$

Hence we find that

$$(p_r)_{2k} - (p_r)_{2k-2} = (-)^f \tfrac{1}{2}m\,(2k+\tfrac{1}{2})\frac{\{2k,\,2f\}}{[\tfrac{1}{2}m,\,2f+1)}\sum_{h=0}^{h=k}(-)^h\frac{\{2k,\,2h\}\,(r,\,2h]}{(\tfrac{1}{2}m,\,2h+1]},$$

$$(p_r)_{2k} = (-)^f \tfrac{1}{2}m \sum_{t=f}^{t=k}(2t+\tfrac{1}{2})\frac{\{2t,\,2f\}}{[\tfrac{1}{2}m,\,2f+1)}\sum_{h=0}^{h=t}(-)^h\frac{\{2t,\,2h\}\,(r,\,2h]}{(\tfrac{1}{2}m,\,2h+1]};$$

and therefore

$$(\delta^{2g}p_0)_{2k} = (-)^{f+g}\tfrac{1}{2}m\sum_{t=f}^{t=k}(2t+\tfrac{1}{2})\frac{\{2t,\,2f\}}{[\tfrac{1}{2}m,\,2f+1)}\frac{\{2t,\,2g\}}{(\tfrac{1}{2}m,\,2g+1]}\Theta,$$

where

$$\Theta = F\{-t+g,\,g+t+\tfrac{1}{2},\,\tfrac{1}{2};\;1,\,-\tfrac{1}{2}m+g+1,\,\tfrac{1}{2}m+g+1\}$$

$$= \frac{(\tfrac{1}{2}m,\,2g+1]\,[\tfrac{1}{2}m,\,2t+1)}{(\tfrac{1}{2}m,\,2t+1]\,[\tfrac{1}{2}m,\,2g+1)}.$$

This expression for $(\delta^{2g}p_0)_{2k}$ will be found to be equal to $\lambda_{2f,\,2g}$ as given by (132) of the present paper, so that the formula in "Fitting" agrees with (142).